SUPPLY NETWORK FORMATION AND FRAGILITY

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Abstract. We model the production of complex goods in a large supply network. Each firm sources several essential inputs through relationships with other firms. Individual supply relationships are at risk of idiosyncratic failure, which threatens to disrupt production. To protect against this, firms multisource inputs and strategically invest to make relationships stronger, trading off the cost of investment against the benefits of increased robustness. A supply network is called fragile if aggregate output is very sensitive to small aggregate shocks. We show that supply networks of intermediate productivity are fragile in equilibrium, even though this is always inefficient. The endogenous configuration of supply networks provides a new channel for the powerful amplification of shocks.

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1. Introduction

Complex supply networks are a central feature of the modern economy. Consider, for instance, a product such as an airplane. It consists of multiple parts, each of which is essential for its production, and many of which are sourced from suppliers. The parts themselves are produced using multiple inputs, and so on.[1] Due to the resulting interdependencies, an idiosyncratic shock can cause cascading failures and disrupt many firms. We develop a theory in which firms insure against supply disruptions by strategically forming supply networks, trading off private gains in the robustness of their production against the cost of maintaining strong supply relationships. Our main results examine how equilibrium supply networks respond to idiosyncratic and aggregate risk. We find that, in equilibrium, (i) the economy is robust to idiosyncratic shocks, yet (ii) small shocks that systemically affect the functioning of supply relationships are massively amplified. Moreover, (iii) the functioning of many unrelated supply chains is highly correlated, and (iv) the complexity of production is key to the nature of these effects and the level of aggregate volatility.

Underlying these results is a discontinuous phase transition in the structure of production networks that arises due to production being complex—reliant on multiple inputs at many stages. Thus, a theoretical contribution of our work is the study of novel equilibrium fragilities in the strategic formation of large networks, and new methods for analyzing them.

Our analysis is built on a model of interfirm sourcing relationships and their disruption by shocks. We now motivate our study of these phenomena with some examples. Firms often rely on particular suppliers to deliver customized inputs. For instance, Rolls-Royce designed and developed its Trent 900 engine for the Airbus A380; Airbus could not just buy the engine it requires off-the-shelf. Such inputs are tailored to meet the customer's specifications, and there are often only a few potential suppliers that a given firm contracts with. Thus, a particular airplane producer is exposed not just to shocks in the overall availability of each needed input, but also to idiosyncratic shocks in the operation of the few particular supply relationships it has formed. Examples of such idiosyncratic shocks include a delay in shipment, a fire at a factory, a misunderstanding by a supplier that delivers an unsuitable component, or a strike by workers. As we have noted, many of a given firm's suppliers themselves will be in a similar position, relying on customized inputs, and so idiosyncratic disruptions to individual relationships and production processes somewhere in a network can have far-reaching effects, causing damage that cascades through the supply chain and affects many downstream firms.[2] On the other hand, many firms multisource key inputs to reduce their dependency on any one supplier and so idiosyncratic disruptions may not cause much reduction in output at all.

Even the basic properties of such complex supply networks are not well-understood. To articulate this, we describe a simple model capturing key aspects of the above examples. There are many products (e.g., airplanes, engines, etc.). Each has many differentiated varieties, produced by small,

[1]For example, an Airbus A380 has millions of parts produced by more than a thousand companies (Slutsken, 2018). In addition to the physical components involved, many steps of production require specific contracts and relationships with logistics firms, business services, etc. to function properly.

[2]Kremer (1993) is a seminal study of some theoretical aspects of such propagation. Carvalho et al. (2020) empirically study how shocks caused by the Great East Japan Earthquake of 2011 propagated through supply networks to locations far from the initial disruption. In Section 6, we discuss further evidence that such disruptions are practically important, and can be very damaging to particular firms.

specialized firms. A given product has a set of customized inputs that must be sourced via supply relationships and are essential to its production—e.g., an airplane requires engines, navigation systems, etc. To source these specific, compatible, varieties firms may have several substitutable sourcing options (i.e., they multisource). Each of a firm's potential supply relationships may operate successfully or not: e.g., one engine manufacturer's delivery may be delayed by a strike, while another is able to deliver normally. In order for a firm to be functional, it must have at least one operating supply relationship to a firm producing each of its essential inputs. To be functional, these producers must in turn satisfy the same condition, and so on—until a point in the supply chain where no customized inputs are required. Our modelling of which supply relationships work is simple: independently, each relationship operates successfully with a probability called the *relationship strength*.[3] This probability represents the chance of avoiding logistical disruptions and failures of contracts in any specific relationship. Given a realization of operating relationships, the firms that are able to produce purchase their required inputs and then sell their products to other firms as well as to consumers. Social welfare is increasing in the number of firms able to produce. Firms make profits from production.

Let us, to begin with, take relationship strength to be exogenous and symmetric across the supply network, and examine aggregate output as we vary this parameter. The key parameters other than relationship strength for describing the mechanics of a supply network are (i) the number of distinct inputs required in each production process; (ii) the number of potential suppliers of each input; and (iii) the depth of supply chains, i.e., the number of steps of specialized sourcing. The first and third dimensions capture the complexity of production, while the second captures the availability of multisourcing. In our model there is a continuum of firms and the fraction of firms functioning is deterministic.[4] This fraction is our main outcome of interest, and we call it the *reliability* of the supply network. Our first main result concerns a distinctive form of sensitivity that can arise in such networks—what we call a precipice. Suppose production is complex—that is, that most firms have multiple essential inputs they need to source, and supply networks are deep, meaning that many steps of such production are needed. We find that there is a discontinuity in reliability as we vary relationship strength, holding all else fixed. When relationship strength falls below a certain threshold (defining the precipice), production drops discontinuously. Thus, if relationship strengths happen to be close to the precipice, a small, systemic, negative shock to relationship strengths is amplified arbitrarily strongly, leading to severe economic damage. We also show that a social planner choosing a level of investments in relationship strengths to maximize welfare would never choose a level such that the supply network is on a precipice.

We give a few practical examples of systemic shocks to relationship strengths. Suppose, first, that the institutions that help uphold contracts and facilitate business transactions suddenly decline in quality, for example due to a political shock. Each supply relationship then becomes more prone to the idiosyncratic disruptions discussed above.[5] Even if the damage to any single relationship

[3]Though the modeling of basic shocks is independent, the interdependence between firms and their suppliers makes failures correlated between firms that (directly or indirectly) transact with each other.

[4]This is by a standard diversification argument. There are enough firms and supply chains operating that none of them is systemically important. On the complementary issue of when individual firms can be systemically important, see Gabaix (2011) and the large ensuing literature.

[5]Blanchard and Kremer (1997) present evidence that the former Soviet Union suffered a large shock of this kind when it transitioned to a market-based economy.

is small (e.g., because usually contracts function without enforcement being relevant), our results show that such a shock can cause widespread disruptions throughout the supply network. For a second concrete interpretation of the kinds of aggregate shocks we have in mind, consider a small shock to the availability of credit for businesses in the supply network. The shock matters for firms that are on the margin between getting and not getting credit that is essential for them to deliver on a commitment. The effect of such a credit shock can be modeled as any given supply relationship being slightly less likely to function (depending on ex ante uncertain realizations of whether a firm is on the relevant margin). Third, during the Covid-19 crisis, there has been much uncertainty about how different supply relationships might be affected, and this can be modeled as a systemic decrease in the probability that suppliers are able to deliver the inputs required from them.

The first result just discussed on exogenous relationship strengths shows that, starting at certain strength levels, supply network functioning *can* be very sensitive to slight shocks to strengths. However, relationship strengths are strategic choices, and the endogenous determination of these is in fact our main focus. Our question is whether a supply network will be near a precipice when relationship strengths are determined by equilibrium choices rather than exogenously or by a planner. Since production is risky, an optimizing firm will strategically choose its relationship strength to manage the risk of its production being disrupted. By investing more, a firm can increase the probability that one of its potential suppliers for each of its essential inputs is able to supply it, hence allowing the firm to produce its output and make profits. Maintaining multiple relationships that facilitate production and provide backup in case of disruption is costly. Firms trade off these costs against the benefits of increased robustness.[6] In our model, firms choose a level of costly investment toward making their supply relationships stronger—i.e., likelier to operate.[7]

Our main findings give conditions for equilibrium relationship strength levels to put the supply network on the precipice, and show that the precipice is *not* a knife-edge outcome. Indeed, we characterize a positive-measure set of parameters (governing the profits of production and the costs of forming relationships) for which the equilibrium supply network is on the precipice. The fragility that a supply network experiences in this regime is highly inefficient: a social planner would never put a supply network on the precipice for the same parameters. As supply networks become large and decentralized one might think that the impact of uncertainty on the probability of successful production would be smoothed by firms' endogenous investments to protect against shocks, and by averaging outcomes across a continuum population. We find the opposite: in equilibrium, there is a very sharp sensitivity of aggregate productivity to relationship strength. This is in contrast to many standard production network models, where the aggregate production function is not too sensitive to small shocks at any outcome. The novelty of our framework comes from the *combination* of two features of production functions that are essential for our results. The first is complexity: The supply networks we study are many layers deep and firms must source multiple essential inputs

[6]Strategic responses to risk in networks is a topic that has attracted considerable attention recently. See, for instance, Bimpikis, Candogan, and Ehsani (2019a), Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011), Talamàs and Vohra (2020), and Erol and Vohra (2018), Amelkin and Vohra (2019), and Acemoglu and Tahbaz-Salehi (2020). On the practical importance of the strength of contracts in supply relationships, see, among others, Antràs (2005) and Acemoglu, Antràs, and Helpman (2007).

[7]This can be interpreted in two ways: (1) investment on the intensive margin, e.g. to anticipate and counteract risks or improve contracts; (2) on the extensive margin, to find more partners out of a set of potential ones.

that cannot be purchased off-the-shelf. The second is the presence of idiosyncratic disruptions to supply. Jointly, these phenomena create the possibility of precipices, which underlie our analysis.

More specifically, our main results concern whether the supply network is on a precipice as we vary an aggregate productivity parameter.[8] Depending on the value of this parameter, the supply network in equilibrium can end up in one of three configurations: (i) a *noncritical equilibrium* where the equilibrium investment is enough to keep relationship strength away from the precipice; (ii) a *critical equilibrium* where equilibrium relationship strength is on the precipice; and (iii) an *unproductive equilibrium* where positive investment cannot be sustained. These regimes are ordered. As the productivity of the supply network decreases from a high to a low level, the regimes occur in the order just given. Each regime occurs for a positive interval of values of the parameter. Equivalently, for an economy consisting of many disjoint supply networks distributed with full support over the parameter space, a positive measure of them will be in the fragile regime, and these will collapse if relationship quality is shocked throughout the economy.

Our analysis makes a conceptual, modeling, and technical contribution to the theory of economic networks. First, we introduce percolation analysis (i.e., disabling some links at random) to an otherwise standard network model of complex production—with *complex* meaning that each firm must source multiple inputs through customized relationships.[9] That combination leads to the finding that under natural assumptions, random failures in a production network result in a *discontinuous* phase transition, where aggregate functionality abruptly disappears when relationship strengths cross a critical threshold. *Simple* production, which does not require risky sourcing of multiple inputs, is not susceptible to this fragility. Second, as a modeling contribution, we demonstrate the tractability of studying equilibrium investments in links (more precisely, investments in the probability that links are operational) in such a setting. By defining a suitable model with a continuous investment choice and a continuum of nodes, investment problems are characterized by relatively tractable first-order conditions, because firms are able to average over the randomness in network realizations. We expect the modeling devices we develop to have other applications. Finally, using our equilibrium conditions to deduce the ordering of regimes discussed above requires developing some new techniques for the analysis of large network formation games.[10] For example, a crucial step in our main results depends on showing that firms' investments in network formation are locally strategic substitutes at the equilibrium investment levels, which depends on subtle properties of equilibrium network structure and incentives that we characterize.

We explore some extensions and implications of our modeling. First, we examine robustness on a number of dimensions. We discuss why the basic insights about fragility extend to alternative specifications of shocks and investment in relationships. Perhaps our most important robustness check concerns relaxing certain symmetries of the supply network that we use to simplify our main analysis. We study how fragility manifests with firm heterogeneity. In particular, we show that precipices continue to obtain in presence of rich heterogeneity across multiple dimensions (number of inputs required, multisourcing possibilities, directed multisourcing efforts, profitability, etc.). One important additional implication of our heterogeneity analysis is that a supply network is only

[8]This parameter can be interpreted as a measure of productivity relative to the costs of maintaining relationships.

[9]A recent model motivated by some of the same questions is Acemoglu and Tahbaz-Salehi (2020), where each node in a production network relies on one failure-prone custom supplier. In Section 7, we discuss related models on information sharing, financial contagion, and other settings.

[10]This differs from and complements the use of graphons in Erol, Parise, and Teytelboym (2020).

as strong as its weakest links: as one product enters the fragile regime, all products that depend on it directly or indirectly are simultaneously pushed into the fragile regime. Second, we interpret our main results in terms of the short- and medium-run resilience of a supply network to shocks, and consider whether the fragility we identify can be ameliorated by natural policy interventions. Third, we show how the supply networks we have studied can be embedded in a larger economy with intersectoral linkages that do not rely on specific sourcing. Our model yields a new channel for the propagation of shocks across sectors, and their stark amplification. Fourth, while the focus of our analysis is on linking complex supply networks to aggregate volatility, we also discuss how the model can provide a perspective on some stylized facts concerning industrial development (Section 8.4). After presenting our results, in Sections 6 and 7 we discuss in detail how they fit into the most closely related literatures.

2. Model: The supply network

Our main object of study is a network whose nodes are a continuum of small firms producing differentiated products. These are connected to their suppliers by a network of potential supply relationships, a random subset of which are realized as operational for sourcing.

An outcome of central interest is the set of firms that is *functional*—i.e., capable of producing. The measure of functional firms will be the key quantity of interest in our model, determining welfare and incentives to invest to form the network, which we introduce in Section 4. The purpose of this section is to set up the basic structure of the production network and define the set of functional firms.

2.1. **Nodes: Varieties of products.** There is a finite set $\mathcal{I}$ of *products*. For each product $i \in \mathcal{I}$, there is a continuum $\mathcal{V}_i$ of *varieties* of i, with a typical variety v being an ordered pair $v = (i, f)$, where $f \in \mathcal{F}_i \subseteq \mathbb{R}$ is a variety index; we take $\mathcal{F}_i = [0, 1/|\mathcal{I}|]$ for all i, so that the total mass of firms (and of varieties) in the supply network is 1. Let $\mathcal{V} = \bigcup_{i\in\mathcal{I}} \mathcal{V}_i$ be the union of all the varieties. These are the nodes in our supply network. Each is associated with a small firm producing the corresponding variety.

2.2. **Links: Potential and realized supply relationships.** First, for each product $i \in \mathcal{I}$ there is a set of required inputs $I(i) \subseteq \mathcal{I}$. Second, each variety $v \in \mathcal{V}$ is associated with a supply chain *depth* $d(v) \in \mathbb{Z}_+$ that specifies how many steps of customized, specifically sourced production are required to produce v, with varieties of larger depth requiring more steps. Different varieties of the same product can have different depths. The measure of varieties with any depth $d \geq 0$ is denoted by $\mu(d)$.

Consider any variety $v \in \mathcal{V}_i$. For each $j \in I(i)$ (i.e., each required input) the variety v has a set of *potential suppliers*, $\mathrm{PS}_j(v) \subseteq \mathcal{V}_j$ and a random subset of *realized suppliers* $\mathrm{S}_j(v) \subseteq \mathrm{PS}_j(v)$.

First consider the varieties $v \in \mathcal{V}$ such that $d(v) = 0$. Specialized sourcing of inputs is not required for these varieties, and they operate without disruption. Thus, in this case, we take $\mathrm{S}_j(v) = \mathcal{V}_j$ for each $j \in I(i)$.

Next, consider any variety $v \in \mathcal{V}_i$ that has depth $d(v) > 0$. For each $j \in I(i)$, the set $\mathrm{PS}_j(v)$ is a finite set of distinct varieties $v' \in \mathcal{V}_j$ with each such v' having depth $d(v') = d(v) - 1$. The identities of these suppliers are independent draws from the set of varieties v' such that $d(v') = d(v) - 1$ (i.e., the set of varieties of compatible depth).[11] Specialized sourcing requirements represent the need for a customized input, the procurement of which is facilitated by relational contracts.

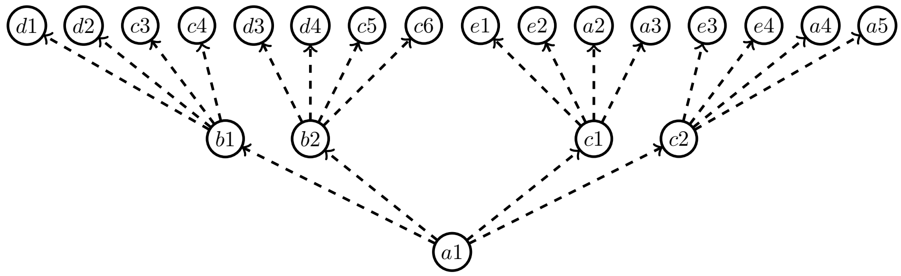


FIGURE 1. Here we consider a potential supply network for variety $(a, 1)$ with underlying products $\mathcal{I} = \{a, b, c, d, e\}$; the relevant input requirements are apparent from the illustration. Each variety requires two distinct inputs. When these inputs must be specifically sourced, there is an edge from the sourcing variety to its potential supplier. We abbreviate $(a, 1)$ as $a1$ (and similarly for other varieties). Here variety $a1$ has depth $d(a1) = 2$. Varieties higher up are upstream of $a1$, and their depths are smaller. Orders or sourcing attempts go in the direction of the arrows, and products are delivered in the opposite direction, downstream.

Each sourcing relationship between v and a variety $v' \in \mathrm{PS}_j(v)$ is *operational* or not—a binary random outcome. For every $v \in \mathcal{V}$, there is a parameter x_v, called *relationship strength* (for now exogenous) which is the probability that any relationship v has with its suppliers is operational. All realizations of relationship operation are independent. The set of actual suppliers $\mathrm{S}_j(v)$ is then obtained by including each potential supplier in $\mathrm{PS}_j(v)$ independently with probability x_v. Whereas the potential supply relationships define compatibilities, the realized supply network identifies which links are actually available for sourcing. The stochastic nature of availability arises, e.g., from uncertainty in delivery of orders, miscommunications about specifications, etc.[12]

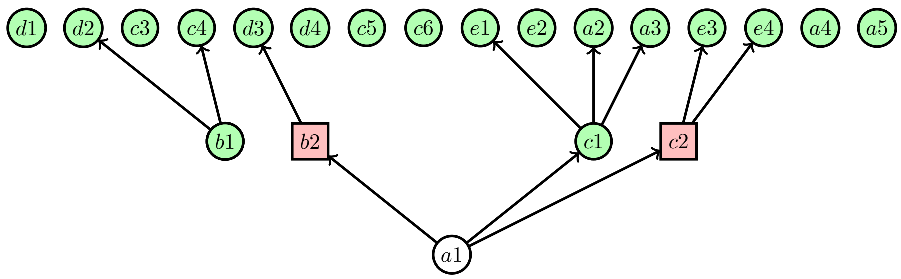
We define two random networks on the set $\mathcal{V}$ of nodes. In the *potential supply network* $\mathcal{G}$, each v has links directed to all its potential suppliers $v' \in \bigcup_{j \in I(i)} \mathrm{PS}_j(v)$. (See Figure 1 for an illustration.) In the *realized supply network* $\mathcal{G}'$, each v has only the operational subset of links, to the realized suppliers: $v' \in \bigcup_{j \in I(i)} \mathrm{S}_j(v)$. See the links in Figure 2 for an illustration of the subset of supply relationships that are operational.

2.3. **The set of functional varieties.** For a given realization of the supply network, we will inductively define which varieties are *functional*, which means that sourcing disruptions do not prevent them from producing.

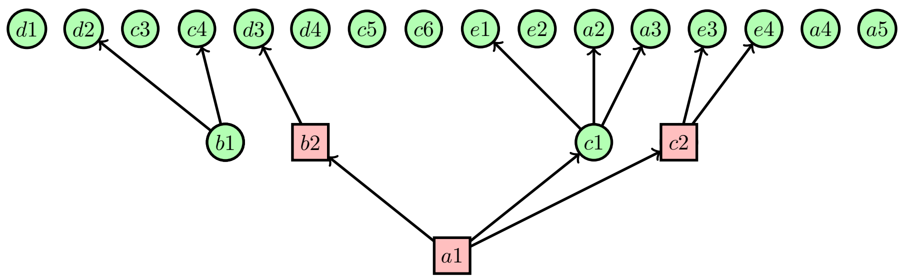
Depth-0 varieties are defined to always be functional: sourcing failures can never prevent the production of such varieties simply because their sourcing is unconstrained by definition. Given functionalities of varieties of depth $d - 1$, a variety v of depth d is functional if and only if its set of realized suppliers $S_j(v)$ contains at least one functional supplier for each input j that v requires. Figure 2 provides an illustration of the potential supply network shown in Figure 1 and a particular realization of operational supply links.

[11]For a formal construction of the potential and realized supply networks, see Appendix A.

[12]In a bit more detail, x_v can capture uncertainty regarding compatibility, whether delivery can happen on time, possible misunderstanding about the required input, access to credit that may be needed to deal with unexpected costs, etc. It will depend on the context or environment in which production occurs, and also (as we explicitly model below) on the investments the firm producing v makes. See Section 6 for more details.



(A) Step 1: Assigning functionalities for depth-1 varieties.



(B) Step 2: Assigning functionalities of depth-2 varieties.

FIGURE 2. An illustration for determining the set of functional varieties given a realized supply network. Functional varieties are represented by green circles, while non-functional ones are pink squares. Varieties that have not yet been assigned to be functional or not are white circles. Varieties of depth 0 are always functional. Panel (A) assigns functionalities to varieties of depth 1. Panel (B) assigns functionalities to varieties of depth 2, making $a1$ nonfunctional.

We let $\mathcal{V}'$ denote the (random) set of functional varieties, and $\mathcal{V}_i'$ denote the set of functional varieties in product i.

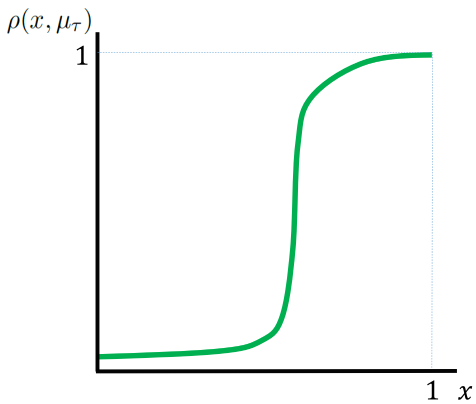
3. Reliability with exogenous relationship strengths

We present our main findings in *regular* supply networks. These are defined by two main symmetries. First, the number of essential inputs is $|I(i)| = m$ for each product i. Second, each variety of depth $d > 0$ has n potential suppliers for each input—i.e. $|\mathrm{PS}_j(v)| = n$ whenever j is one of the required inputs for variety v. Networks with these homogeneous structures are depicted in Figures 1 and 2. In this section, we posit that relationship strengths are given by $x_v = x$, the same number for each v, and this x is exogenous.[13]

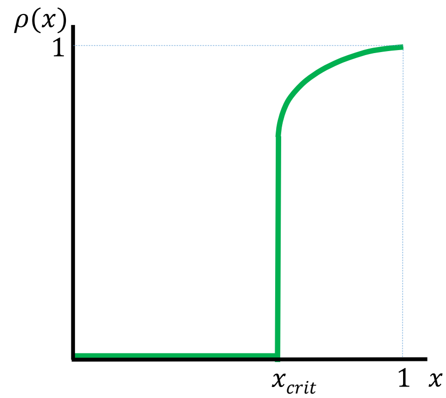
It will be important to characterize how the measure of functional firms $\mathcal{V}'$ varies with relationship strength x. Denote by $\widetilde{\rho}(x, d)$ the probability that an arbitrary variety of depth d is functional when relationship strength is x. Zero-depth varieties are functional by our assumption that they do not need any specialized inputs: $\widetilde{\rho}(x, 0) = 1$. By symmetry of the supply tree, the probability $\widetilde{\rho}(x, d)$ does indeed only depend on x and d (and we calculate it explicitly below in Section 3.1.1). The probability that a variety selected uniformly at random is functional is called the *reliability* of the supply network. Since it depends on the distribution of depths μ, we denote this by $\rho(x, \mu)$ and define it as

$$\rho(x, \mu) = \sum_{d=0}^{\infty} \mu(d)\widetilde{\rho}(x, d). \tag{1}$$

[13]We endogenize relationship strengths in the next section, and we relax the symmetry assumptions in Section 6.3.2.



(A)



(B)

FIGURE 3. Panel (A) shows how reliability varies with relationship strength x for a particular τ. Panel (B) depicts a correspondence that is the limit of the graphs $\rho(x, \mu_\tau)$ as τ tends to infinity.

Deep supply networks: Taking limits. A focus throughout will be the case where a typical variety has large depth.[14] We thus introduce a notation for asymptotics: we fix a sequence $(\mu_\tau)_{\tau=1}^{\infty}$ of distributions, τ is a parameter we will take to be large. We assume μ_τ places probability at least $1 - \frac{1}{\tau}$ on $[c\tau, \infty)$ for some $c > 0$.[15] For instance, we can take μ_τ to be the geometric distribution with mean τ (in which case τ has an exact interpretation as average depth). For large τ, if inputs are single-sourced ($n = 1$) and links fail with positive probability, there will only be a very remote probability of successful production. We therefore restrict attention to the case of multisourcing ($n \geq 2$).

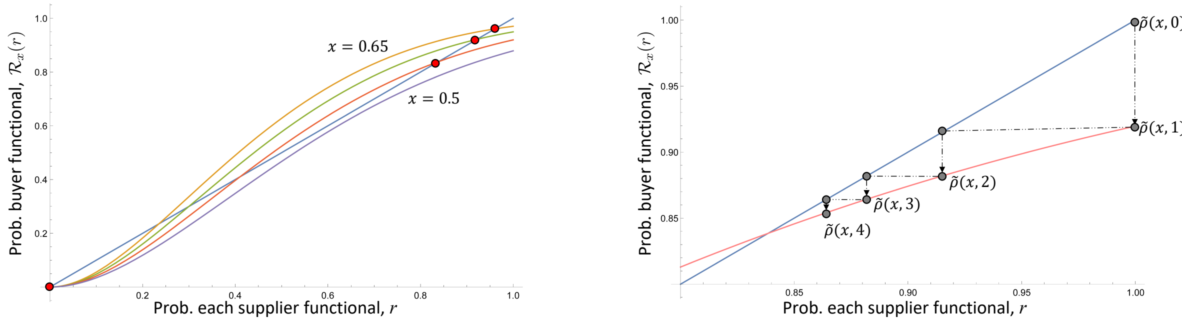
3.1. **A discontinuity in reliability.** A key implication of the model is the shape of the aggregate reliability function as we vary x. As we will see in the next section, this function is closely linked to the aggregate production function of the supply network, and thus its shape underlies many of our results. Our first result characterizes important properties of this shape.

Proposition 1. Fix any $n \geq 2$ and $m \geq 2$. Then there exist positive numbers $x_{\text{crit}}, \overline{r}_{\text{crit}} > 0$ such that,

(i) if $x < x_{\text{crit}}$, we have that $\rho(x, \mu_\tau) \to 0$ as $\tau \to \infty$. That is, reliability converges to 0.
(ii) if $x > x_{\text{crit}}$, then, for all large enough τ, we have $\rho(x, \mu_\tau) > \overline{r}_{\text{crit}}$. That is, reliability remains bounded away from 0.

In Figure 3(a) we plot the reliability function $\rho(x, \mu_\tau)$ for a fixed finite value of τ against the probability x of each relationship being operational. One can see a sharp transition in relationship strength x. This can be seen more sharply in Figure 3(b), where we plot the limit of the graph shown in (a) as $\tau \to \infty$. We use the $m = n = 2$ case here as in our illustrations above. There is a critical value of relationship strength, which we call x_{crit}, such that the probability of successful production is 0 when $x < x_{\text{crit}}$, but then increases sharply to more than 70% for all $x > x_{\text{crit}}$. Moreover, the

[14]Supplementary Appendix SA4 investigates how reliability varies with investment in production trees with bounded depth.
[15]Note that this means varieties of low depth have many incoming edges in the potential supply network, since there are relatively few of them, but they make up a relatively large number of the nodes in a typical production tree.



(A) (B)

FIGURE 4. Panel (A) shows the probability, $\mathcal{R}_x(r)$, that a focal firm is functional as a function of r, the probability that a random supplier is functional. Here we use the parameters $n = 4$, $m = 2$ and $x \in \{0.5, 0.55, 0.6, 0.65\}$. The intersections with the 45 degree line marked by the red circles represent reliability values (for deep supply trees) associated with the given exogenous parameters. Panel (B) shows, for a given value of x (namely, $x = 0.55$), the first four values of $\widetilde{\rho}(x, d)$ given by equation (2). Note that the intersection is approached quite closely after a few steps.

derivative of the limit reliability graph as we approach x_{crit} from above grows arbitrarily large (i.e., $\lim_{x \downarrow x_{\text{crit}}} \lim_{\tau \to \infty} \rho'(x, \mu_\tau) = \infty$). This has important ramifications, as we will see. An immediate one is that small improvements in relationship strength x, for example through the improvement of institutions, can have large payoffs for an economy, and the net marginal returns on investment in x can change sharply from being negative to being positive and very large.

3.1.1. *The reasons for the shape of the reliability function.* To explain the logic behind the proposition, let us now calculate the probability that a given variety v with depth d is functional. Recall that we denote by $\widetilde{\rho}(x, d)$ the probability that a variety of depth d is functional when relationship strength is x. We will argue that this can be expressed recursively as follows. First, $\widetilde{\rho}(x, 0) = 1$, since varieties of depth 0 are sure to be functional. Then, for a suitably defined function $\mathcal{R}_x : [0, 1] \to [0, 1]$, we can write the depth-d reliability in terms of the depth-$(d-1)$ reliability:

$$\widetilde{\rho}(x, d) = \mathcal{R}_x(\widetilde{\rho}(x; d-1)). \tag{2}$$

Indeed, more explicitly, the function that makes this true is[16]

$$\mathcal{R}_x(r) = (1 - (1 - xr)^n)^m.$$

As we look at networks with large typical depths, it is the $\widetilde{\rho}(x, d)$ for large d that will matter. Proposition 6 in Appendix B.1 shows that there is a unique correspondence $\rho(x)$ that is the limit of the graphs of $\rho(x, \mu_\tau)$ in a suitable sense as $\tau \to \infty$. By analyzing this correspondence we can show that a sharp transition like that shown in Figure 3(B) occurs for any complexity $m \geq 2$ and any multisourcing level $n \geq 2$, once depths become large.

The intuition for the sharp transition is illustrated in Figure 4. In panel (A), we plot the probability that a given firm is functional against the probability that its suppliers are functional

[16]Fix a variety and consider any one of its inputs. For a given supplier of that input, by definition its reliability is the argument r, and the probability that the link to the supplier is operational is x. The probability that both events happen is xr. The probability that this combination of events happens for at least one of the n potential suppliers of the first input is therefore $1 - (1 - xr)^n$. Finally, the probability that for all m inputs, such a "good event" happens is $(1 - (1 - xr)^n)^m$.

(which is taken to be common across the suppliers, and denoted by r). The curve $\mathcal{R}_x(r)$ is shown for several values of x. In panel (B), one of these $\mathcal{R}_x(r)$ curves is sketched (note the change in axes) along with the 45-degree line; the figure shows that if we repeatedly apply formula (2) starting with $\widetilde{\rho}(x, 0) = 1$, we get a a sequence of $\widetilde{\rho}(x, d)$ converging to the largest fixed point of $\mathcal{R}_x(r)$. As supply networks become deep, a firm and its suppliers occupy essentially equivalent positions, so it is natural that the limit probability of being functional for large-depth firms is equal to such a fixed point. The reliability levels r that are fixed points of the function $\mathcal{R}_x(r)$ in equation (2) are given by its intersections with the 45-degree line, illustrated in panel (A) for several values of x (each corresponding to one of the curves). We will now examine how the largest such intersection depends on x. For high enough x, the function $\mathcal{R}_x(r)$ has a fixed point with $r > 0$. When x is below a certain critical value x_{crit}, the graph of $\mathcal{R}_x$ has no nonzero intersection with the 45-degree line and so the limit of $\widetilde{\rho}(x, d)$ as $d \to \infty$ is 0. Crucially, the largest fixed point of $\mathcal{R}_x(r)$ does not decrease continuously down to 0 as we lower x. Instead, it drops down discontinuously when x decreases past x_{crit}—defined as the (positive) value of x where there is a point of tangency between $\mathcal{R}_x(r)$ and the 45-degree line. At this point, the largest r solving $r = \mathcal{R}_x(r)$ jumps down discontinuously from the r corresponding to this point of tangency to 0. This is the discontinuous "precipice" drop. As we explain in Section 6.1.1 where we contrast complex production ($m > 1$) with simple production ($m = 1$), the convex-then-concave shape of the $\mathcal{R}_x$ curve is essential for creating a precipice.

3.1.2. *Comparative statics of the reliability function.* Some straightforward comparative statics can be deduced from what we have said. If n (multisourcing) increases while all other parameters are held fixed, then one can check that $\mathcal{R}_x$ (as illustrated in Figure 4(A)) increases pointwise on $(0, 1)$, and this implies that all the $\widetilde{\rho}(x, d)$ increase. It follows that the ρ curve moves upward, and the discontinuity occurs at a lower value of x.

Similarly, when m (complexity) increases, the $\mathcal{R}_x$ curve decreases pointwise, implying that all the $\widetilde{\rho}(x, d)$ decrease. It follows that the ρ curve moves downward, and the discontinuity occurs at a higher value of x.

4. Supply networks with endogenous relationship strength

In this section, we study the endogenous determination of relationship strength. We first study a planner's problem—choosing an optimal value of strength for all relationships. We then turn to a decentralized problem in which firms invest in their own relationship strengths. Throughout, we focus on symmetric outcomes.

Our main results show that while investments that put the supply network on the precipice are very inefficient, they need not be knife-edge or even unlikely outcomes in equilibrium.

To connect the graph-theoretic notion of reliability in the previous section with economic outcomes, we introduce a relationship between reliability and output. In particular, we posit that there is a function Y such that $Y(\mathcal{V}')$ is the (expected) aggregate gross output associated with a given set $\mathcal{V}'$ of functional firms. We assume that it satisfies the following assumption:

Property A. In a regular supply network, $Y(\mathcal{V}') = h(\rho(x, \mu))$, where $h : [0, 1] \to \mathbb{R}$ is a strictly increasing, concave function with bounded and continuous derivative. Moreover, $h(0) = 0$.

In Appendix C, we provide a microfoundation for this assumption: we define a standard production network model on top of the supply network, being explicit about production functions and markets at the micro level. There, we show that aggregate output satisfies Property A. However, our main results—about efficient and equilibrium networks—do not rely on any details of the microfoundations. Instead, they rely only on assumptions about benefits and costs of production that we bring out into named properties.

Property A sets the stage for our study of the planner's problem.

4.1. **A planner's problem.** We study a planner who chooses a global x that determines the values of all relationship strengths, $x_v = x$. This can be interpreted as investing in the quality of institutions, at a cost that we will introduce below. As stipulated in Property A, the gross output of the supply network is an increasing function of reliability, $Y(\mathcal{V}') = h(\rho(x, \mu))$, where reliability ρ depends on the symmetric level of relationship strengths x and depth distribution μ. The planner's cost of a given choice of x enters through subtracting a quantity $\frac{1}{\kappa} c_P(x)$ of output, where κ is a strictly positive parameter and $c_P : [0,1] \to \mathbb{R}$ is a fixed function. Higher values of κ have the interpretation that they shift down the costs of obtaining a given level of relationship strengths, and hence a given level of gross output. The planner seeks to maximize expected net aggregate output by choosing relationship strengths x, and hence solves the **planner's problem**

$$\max_{x \in [0,1]} h(\rho(x, \mu)) - \frac{1}{\kappa} c_P(x). \tag{3}$$

We make the following assumption concerning $c_P(x)$:

Property B. The function $c_P : [0,1] \to \mathbb{R}$ is continuously differentiable and weakly convex, with $c_P(0) = 0$, $c_P(x_{\text{crit}}) > 0$, $c_P'(0) = 0$, and $\lim_{x \to 1} c_P'(x) = \infty$.

Substantively, the conditions in this property entail that there are increasing marginal social costs of relationship strength and achieving the critical level of reliability requires a positive investment. The Inada conditions on derivatives ensure good behavior of optima.

Define the correspondence

$$x^{SP}(\kappa, \mu) := \operatorname*{argmax}_{x \in [0,1]} h(\rho(x, \mu)) - \frac{1}{\kappa} c_P(x).$$

This gives the values of x that solve the social planner's problem for a given κ and distribution of supply chain depths μ. As elsewhere, we consider a sequence $(\mu_\tau)_{\tau=1}^{\infty}$ of depth distributions, where μ_τ places mass at least $1 - \frac{1}{\tau}$ on $[\tau, \infty)$.

Proposition 2. Fix any $n \geq 2$ and $m \geq 2$. Then there exists a number $\kappa_{\text{crit}} > 0$ such that, for some $\epsilon > 0$, the following statements hold for all large enough τ:

(i) for all $\kappa < \kappa_{\text{crit}}$, all values of $x^{SP}(\kappa, \mu_\tau)$ are at most $x_{\text{crit}} - \epsilon$, have cost less than ϵ, and yield reliability less than ϵ;

(ii) for all $\kappa > \kappa_{\text{crit}}$, all values of $x^{SP}(\kappa, \mu_\tau)$ are at least $x_{\text{crit}} + \epsilon$ and yield reliability at least $r_{\text{crit}} + \epsilon$;

(iii) for $\kappa = \kappa_{\text{crit}}$, all values of $x^{SP}(\kappa, \mu_\tau)$ are outside the interval $[x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$ and reliability is outside the interval $[\epsilon, r_{\text{crit}} + \epsilon]$.

The first part of Proposition 2 says that when κ is sufficiently low, it is too costly for the social planner to invest anything in the quality of institutions, and hence reliability is very low. As κ crosses a threshold κ_{crit}, it first becomes optimal to invest in institutional quality. At this threshold, the social planner's investment increases discontinuously. Moreover, it immediately increases to a level strictly *above* x_{crit}; and for all larger κ all solutions stay above x_{crit}.

It is worth emphasizing that the planner never chooses to invest near the critical level x_{crit}. The reason is as follows: sufficiently close to $x = x_{\text{crit}}$, the marginal social benefits of investing grow arbitrarily large in the limit as τ gets large while marginal costs at x_{crit} are bounded, and so the social planner can always do better by increasing investment at least a little. In contrast, in Section 5 we will see that individual investment choices can put the supply network on the precipice in equilibrium, and this is not a knife-edge scenario.

4.2. **Decentralized investment in relationship strengths.**

4.2.1. *Setup and timing.* Now we formulate a simple, symmetric, model of decentralized choices of relationship strengths. The decision-makers in this richer model are firms. In each product i, there is a continuum of separate firms (i, f), where $f \in [0, \frac{1}{|\mathcal{I}|}]$. The firm (i, f) owns the corresponding variety, $v = (i, f)$; our notation identifies a firm with its variety. We often abbreviate both by if.

Firms simultaneously choose *investment levels* $y_{if} \geq 0$. Choosing a level y_{if} has a private cost $c(y_{if})$. The random realization of the supply network occurs after the firm chooses its investment level.[17] If a firm chooses an investment level y_{if}, then all sourcing links from its variety (i, f) have relationship strength

$$x_{if} = \underline{x} + y_{if}.$$

The intercept $\underline{x} \geq 0$ is a baseline probability of relationship operation that occurs absent any costly investment. This can be interpreted as a measure of the quality of institutions—e.g., how likely a "basic contract" is to deliver.[18] The main purpose of this baseline level is as a simple channel to systemically shock relationship strengths throughout the supply network.

The timing is as follows.

1. Firms simultaneously choose their investment levels.
2. The realized supply network is drawn and payoffs are enjoyed.

An *outcome* is given by relationship strengths x_{if} for all firms if. An outcome is symmetric if all firms have the same relationship strength: $x_{if} = x$ for all if.

4.2.2. *Payoffs and equilibrium.* A firm's payoff at an outcome can be written as

$$u_{if} = G_{if} - \frac{1}{\kappa} c(x_{if} - \underline{x}).$$

This is the firm's expected gross profit, G_{if}, minus the cost of its investment, $y_{if} = x_{if} - \underline{x}$, in relationship strength. We will now discuss the parts of this payoff function in turn.

[17]The supply network realization is defined as an assignment of depths to all varieties, and the graphs $\mathcal{G}$ and $\mathcal{G}'$ from Section 2. The assumption that investments are made before this realization is technically convenient, as it keeps the solution of the model symmetric. For example, a firm knows that after some number of stages of production, disruption-prone contracts will not be needed by its indirect suppliers (e.g., because these suppliers are able to use generic inputs or rely on inventories). However, the firm does not know how many steps this will take. See Section 6.3.1 for an extension where firms have some information about their depths.

[18]A natural interpretation is that this is a feature of the contracting environment—concretely, for instance, it could reflect the quality of the commercial courts.

We begin with the firm's costs. We assume that they satisfy the following property, where $\kappa > 0$ is a parameter:

Property B′. A firm's cost is given by $\frac{1}{\kappa}c(x_{if} - \underline{x})$, where the following conditions hold:

(i) $\underline{x} < x_{\text{crit}}$;
(ii) c' is increasing, continuously differentiable, and strictly convex, with $c(0) = 0$;
(iii) the Inada conditions hold: $\lim_{y \downarrow 0} c'(y) = 0$ and $\lim_{y \uparrow 1-\underline{x}} c'(y) = \infty$.

The first part of this assumption ensures that baseline relationship strength is not so high that the supply network is guaranteed to be productive even without any investment. The second part imposes assumptions on investment costs that ensure agents' optimization problems are well-behaved. The Inada conditions, as usual, ensure that investments are interior. Here κ plays the same role as it did in our social planner optimization exercise: scaling down the costs of investing in relationship strength—and hence achieving a given level of productivity.

We now turn to specifying gross profits at a given outcome. Because we will characterize symmetric equilibria, we need to specify the gross profits of a given firm only for symmetric behavior by other firms. Firms make no gross profits conditional on not producing, and their profits conditional on producing satisfy the following assumption.

Property C. At a symmetric outcome with reliability r, conditional on being functional, a firm makes gross profits $g(r)$, where $g : [0, 1] \to \mathbb{R}_+$ is a decreasing, continuously differentiable function.

Property C requires that profits are higher when fewer firms are functioning and there is less competition. Appendix C microfounds this property in the same production network model that we used to microfound Property A.

Let $P(x_{if}; x, \mu)$ be the probability with which a firm if is functional if firm if's relationship strength is x_{if} and all other firms choose symmetric relationship strengths x inducing reliability r. Then, under Property C, we have that $G_{if} = P(x_{if}; x, \mu)g(r)$. Thus, recalling the payoff formula given at the start of this section, the net expected profit of firm if when it has relationship strength x_{if} and all other firms have relationships strengths x, resulting in reliability r, is

$$\Pi_{if} = P(x_{if}; x, \mu)g(r) - \frac{1}{\kappa}c(x_{if} - \underline{x}). \tag{4}$$

Finally, because welfare properties of equilibria will play a role in our analysis, we define social welfare (for symmetric outcomes). The gross output of production given reliability r has a value of $h(r)$, as in the previous section on the planner's problem. (Some of this goes to gross profits and some to consumer welfare, but the sum is given by $h(r)$.) The planner's cost function is simply the total of firms' costs,

$$\sum_{i \in \mathcal{I}} \int_{f \in \mathcal{F}_i} \frac{1}{\kappa}c(x - \underline{x})df = \frac{1}{\kappa}c(x - \underline{x}),$$

where we have used our assumption that the mass of firms in each industry is $1/|\mathcal{I}|$, so that the total mass of firms is 1. Thus, if x is the relationship strength of all firms, the social welfare function is[19]

$$\text{Welfare} = h(\rho(x, \mu)) - \frac{1}{\kappa}c(x - \underline{x}). \tag{5}$$

[19]This coincides with the welfare function of the previous section if we set $c_P(x) = c(x - \underline{x})$ for $x \geq \underline{x}$, and $c_P(x) = 0$ otherwise.

5. Equilibrium supply networks and their fragility

We now study the equilibrium of our model: its productivity and its robustness. This section builds up to a main result: Theorem 1. We show that in the limit as production networks become deep, there are three regimes. First, for low values of the parameter κ, there is an unproductive regime in which equilibrium reliability is arbitrarily low. Next, for intermediate values of κ, there is a critical regime in which equilibrium relationship strengths are very close to x_{crit} and arbitrarily small shocks to relationship strength lead to discontinuous drops in production. Finally, there is a noncritical regime in which equilibrium relationship strengths are above x_{crit} and the supply network is robust to small shocks.

It is worth writing (4) more explicitly. To do this, we calculate $P(x_{if}; x, \mu)$, the probability with which if is able to produce, as a function of firm if's relationship strength x_{if} given that all other firms choose a symmetric relationship strength x:[20]

$$P(x_{if}; x, \mu) := \mu(0) + \mathbb{E}\left[1 - (1 - x_{if}\widetilde{\rho}(x, d-1))^n\right]^m,$$

where d inside the expectation is drawn from the depth distribution μ conditional on depth being at least 1. Recall from equation (2) in Section 3.1.1 the formula for $\widetilde{\rho}$.

Definition 1. We say $x \geq \underline{x}$ is a *symmetric equilibrium* if $x_{if} = x$ maximizes $\Pi_{if}(x_{if}; x, \mu)$ defined in (4) for any firm if. It is a *symmetric undominated equilibrium* if it is a symmetric equilibrium and, among symmetric equilibria, maximizes the social welfare defined in (5).

When we refer to an equilibrium in the sequel, we mean a symmetric undominated equilibrium unless otherwise noted. Note that a symmetric equilibrium is defined by the level of relationship strength $x = \underline{x} + y$ realized in it, rather than the level of investment y. This turns out to be more convenient.

Our equilibrium definition requires that all firms' investment choices are equal and are mutual best responses to each other. Lemma 5 in the appendix shows that the efficiency condition selects the symmetric equilibrium associated with the highest investment level, and hence highest reliability. Note that it is always a best response for a firm to choose zero investment when all others choose zero investment. Our equilibrium definition abstracts from potential miscoordination on the zero investment level, or other inefficient ones, by selecting the symmetric equilibrium that maximizes welfare.

In the limit, as the expected depth of the supply networks becomes large, if firms symmetrically choose investments $y_{if} = 0$ then the reliability is $\rho(\underline{x}) = 0$ as $\underline{x} < x_{\text{crit}}$ (by Property B$'$). Hence, for large enough τ, $x_{if} = \underline{x}$ maximizes $\Pi_{if}(x_{if}; \underline{x}, \mu_\tau)$ and so there always exists an equilibrium.

In analyzing the symmetric equilibria it is helpful to make an assumption on the environment that ensures that the first-order conditions of firms' problems are sufficient for optimality among interior solutions. We first state the assumption and then formulate a condition on primitives that is sufficient for it to hold.

Assumption 1. For any τ and any $x > x_{\text{crit}}$ the function $x_{if} \mapsto \Pi_{if}(x_{if}; x, \mu_\tau)$ has a unique interior local maximum for all large enough τ.[21]

[20]Note that because there is a continuum of firms the probability that a firm appears in its potential supply network upstream of itself is 0. Thus the reliability of if's suppliers does not depend on x_{if}.

[21]The assumption permits another local maximum at a corner. We rule this out separately.

Assumption 1 will be maintained in the sequel, along with Properties A, B′, and C. The following lemma shows that we may always set $\underline{x}$ so that Assumption 1 is satisfied.

Lemma 1. For any $m \geq 2$, and any $n \geq 2$, there is a number[22] $\widehat{x}$, depending only on m and n, such that, for large enough τ we have: (i) $\widehat{x} < x_{\text{crit}}$; and (ii) if $\underline{x} \geq \widehat{x}$, then Assumption 1 is satisfied.

To see why this lemma implies that Assumption 1 is satisfied for a suitable choice of $\underline{x}$, consider any environment where $\underline{x} \in [\widehat{x}, x_{\text{crit}})$. Part (i) of the lemma guarantees that the interval $[\widehat{x}, x_{\text{crit}})$ is nonempty, and part (ii) guarantees that Assumption 1 is satisfied for values of $\underline{x}$ in this range.[23] In other words, the lemma guarantees that there is an interval of possible baseline levels of robustness which are short of the critical level (so that fragility is not ruled out a priori) but high enough to ensure that the firms' maximization problem is amenable to a first-order approach.

We now characterize the equilibrium behavior.

Theorem 1. Fix any $n \geq 2$, $m \geq 3$, and $\epsilon > 0$.[24] There are thresholds $\underline{\kappa}(n,m) < \overline{\kappa}(n,m)$ and a threshold $\underline{\tau}$, so that for all $\tau \geq \underline{\tau}$ there is a unique symmetric undominated equilibrium, with a relationship strength denoted by $x^*_\tau(\kappa)$, satisfying the following properties:

(i) If $\kappa < \underline{\kappa}$, there is no investment: $x^*_\tau(\kappa) = \underline{x}$.
(ii) For $\kappa \in [\underline{\kappa}, \overline{\kappa}]$, the equilibrium relationship strength satisfies $x^*_\tau(\kappa) \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$. We call such equilibria *critical.*
(iii) For $\kappa > \overline{\kappa}$, the equilibrium relationship strength satisfies $x^*_\tau(\kappa) > x_{\text{crit}}$. We call such equilibria *non-critical.*

Moreover, for $\tau \geq \underline{\tau}$, the function $x^*_\tau(\kappa)$ is increasing on the domain $\kappa > \underline{\kappa}$.

If we think of different supply networks as being parameterized by different values of κ in a compact set, Theorem 1 implies that in the limit as τ gets large, there will be a positive fraction of supply networks in which firms will choose relationship strengths converging to x_{crit} in equilibrium. This contrasts with the social planner's solution, which never selects relationship strengths near x_{crit}. It also means that a positive fraction of supply networks end up perched on the precipice, vulnerable to small unanticipated systemic shocks. This vulnerability extends to anticipated shocks, as discussed in Section 6.2.2.

Figure 5 helps give some intuition for Theorem 1. In any symmetric equilibrium, the reliability of each firm must be consistent with the symmetric investment level x chosen by all other firms—we must be somewhere on the reliability curve we derived in Section 2. The graphs labeled $\rho(x)$ in panels (A)–(D) of Figure 5 illustrate the shape of the reliability curve for large τ. Further, in any symmetric equilibrium each firm's (symmetric) investment choice of x must be a best response to the reliability of its suppliers. The curves labeled $BR(x)$ in panels (A)–(D) depict the best-response function; these curves should be thought of as having their independent variable (others' reliability) on the vertical axis, and the best-response investment on the horizontal axis. The panels show the best-response curves for increasing values of κ. Intersections of these two curves

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[22] In the proof, we give an explicit description of $\widehat{x}$ in terms of the shape of the function $P(x_{if}; x, \mu_\tau)$.
[23] While conditions we identify in Lemma 1 are sufficient for satisfying Assumption 1, they are not necessary.
[24] In this result, we restrict attention to the case of $m \geq 3$. It is essential for our results that supply networks are complex ($m \geq 2$), but the case of $m = 2$ generates some technical difficulties for our proof technique so we consider $m \geq 3$. In numerical exercises, our conclusions seem to also hold for $m = 2$.

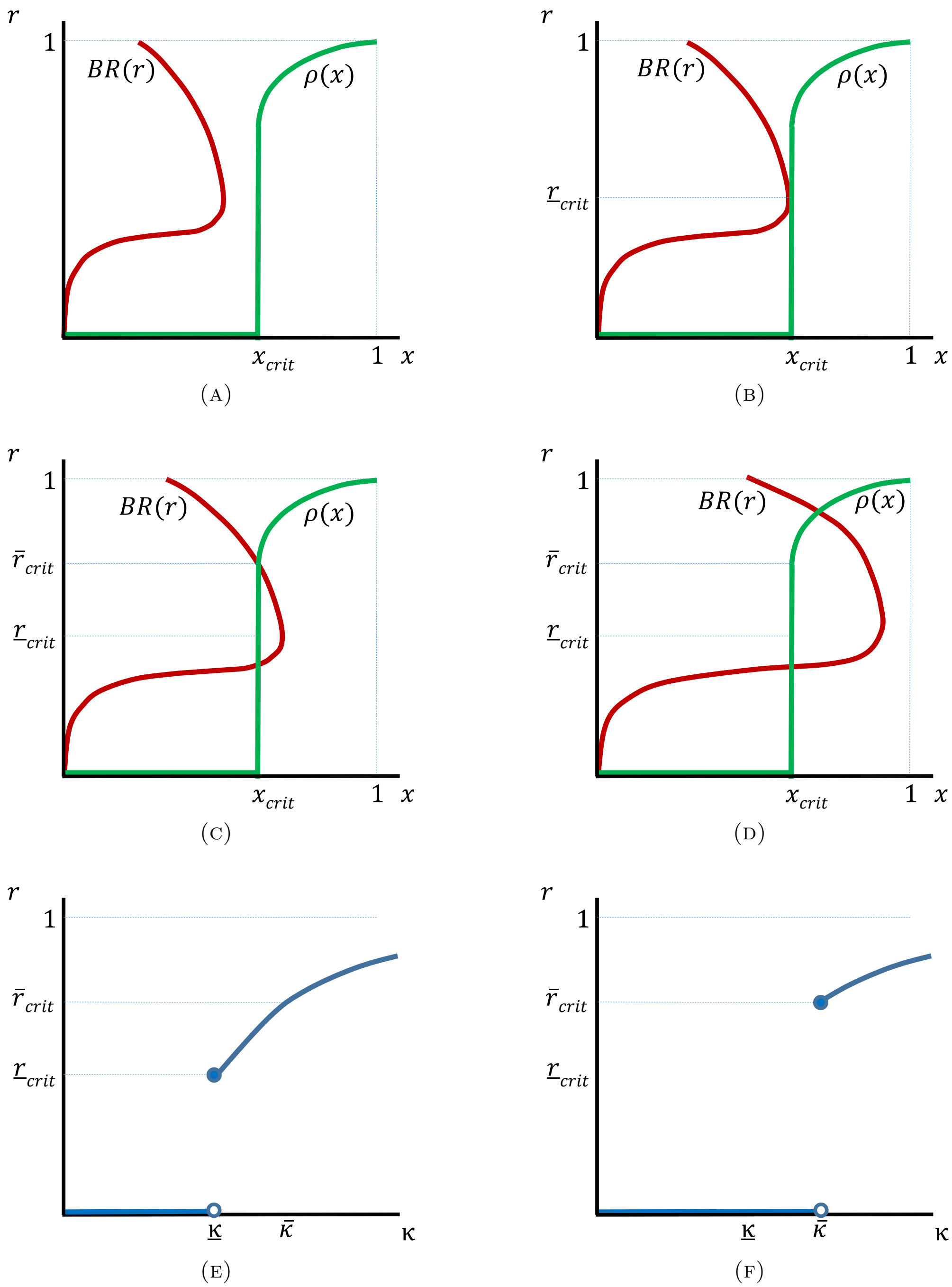


FIGURE 5. Panel (A) shows an equilibrium for $\kappa < \underline{\kappa}$. Panel (B) shows an equilibrium with $\kappa = \underline{\kappa}$. Panel (C) shows an equilibrium with $\kappa = \overline{\kappa}$. Panel (D) shows an equilibrium with $\kappa > \overline{\kappa}$. Panel (E) plots how equilibrium reliability varies with κ. Panel (F) shows reliability following an arbitrarily small negative shock to institutional quality $\underline{x}$ as κ varies: note the drop for $\kappa \in [\underline{\kappa}, \overline{\kappa}]$

correspond to potential symmetric equilibria. When there are multiple intersections, we focus on the one associated with the highest reliability, which is the one that is selected by Definition 1. Thus equilibrium reliability is 0 for κ sufficiently small, jumps up discontinuously to $\underline{r}_{\text{crit}}$ at $\underline{\kappa}$, and is increasing in κ thereafter. This is shown in panel (E). Reliability clearly increases as κ ranges over the interval $[\underline{\kappa}, \overline{\kappa}]$. Nevertheless, for κ in this interval, equilibrium investment for large

τ remains arbitrarily close to x_{crit}, since the equilibrium is on the essentially vertical part of the reliability curve. In other words, equilibrium investment choices bunch around x_{crit} for an open interval of values of κ. For all $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ a slight shock causing relationship strengths to diminish from $\underline{x}$ to $\underline{x} - \epsilon$ causes relationship strengths to fall below x_{crit}, and makes equilibrium production collapse. Panel (F) shows reliability after such a shock for different values of κ. As can be seen by comparing panels (E) and (F), the shock does not change the values of reliability at κ below $\underline{\kappa}$ or above $\overline{\kappa}$: the curves in (E) and (F) are essentially identical in those ranges. However, for $\kappa \in [\underline{\kappa}, \overline{\kappa}]$, reliability drops discontinuously to 0.

We now pause to comment on the techniques used to establish these results. While the intuition looks straightforward in our sketches, proving Theorem 1 is technically challenging. We have to get a handle on the shape of the best-response correspondence, and show that the highest intersection of it with the reliability curve moves as depicted in Figure 5. Investment incentives (which determine the shape of the best-response curve, and the implications of changing κ) are complex, and so equilibrium investment is difficult to characterize directly. We do not have an explicit expression for the best-response curve. It also does not have any global monotonicity properties that might permit standard comparative statics approaches. Investments in relationship strength are strategic complements in some regions of the parameter space, and strategic substitutes in others.[25] However, some control on the slopes of the intersecting curves is necessary to analyze the dependence of the intersection on parameters. A key step in our analysis is showing that if the highest intersection of best response and reliability curves is at $r \geq r_{\text{crit}}$, then the best response curve must be negatively sloped in r there, and strictly so for an intersection at $r > r_{\text{crit}}$. We do so by establishing certain properties of the system of polynomial equations that (asymptotically) characterizes the intersection. Another cruicial step is to show that there is at most one point of intersection above $\overline{r}_{\text{crit}}$ in our diagrams. This permits us to focus on a unique outcome of interest, and to sign the effects of moving κ unambiguously, which is important for the results on equilibrium fragility.

At a technical level, there are two parts to many of our proofs. One part focuses on an idealized "limit supply network," which, in a suitable sense, has infinite depth and therefore a lot of symmetry (a firm's suppliers look exactly like the firm itself). This symmetric network gives rise to expressions that we can manipulate to establish the key facts mentioned above. There is then a separate task of transforming these limit results into statements about the supply network with large but finite depths.

Our next result, Corollary 1, implies that the comparative statics of equilibrium as the baseline quality of institutions $\underline{x}$ changes are analogous to those documented with respect to κ in Theorem 1. Here we explicitly include $\underline{x}$ as an argument in x^*.

Corollary 1. Suppose $\kappa' > \kappa$. Then, for large enough τ, if $x^*_\tau(\kappa', \underline{x}) > x^*_\tau(\kappa, \underline{x})$, there is an $\underline{x}' > \underline{x}$ such that $x^*_\tau(\kappa', \underline{x}) = x^*_\tau(\kappa, \underline{x}')$.

[25]When a firm's suppliers are very unreliable, there is little incentive to invest in stronger relationships with them—there is no point in having a working supply relationship when the suppliers cannot produce their goods. On the other hand, when a firm's suppliers are extremely reliable, a firm can free-ride on this reliability and invest relatively little in its relationships, knowing that as long as it has one working relationship for each input it requires, it is very likely to be able to source that input.

5.1. **Fragility.** Critical equilibria are important because, as the example in Figure 5 shows, they create the possibility of fragility: small unanticipated shocks to relationship strengths via a reduction in $\underline{x}$ can result in a collapse of production. We formalize this idea by explicitly examining how the supply network responds to a shock to the baseline quality of institutions $\underline{x}$, which for simplicity is taken to have zero probability. (The analysis is robust to anticipated shocks that happen with positive probability—see Section 6.2.2.)

Definition 2 (Equilibrium fragility)**.**

- There is *equilibrium fragility* at κ if for any $\eta, \epsilon > 0$, for large enough τ, we have

$$\rho(x^*_\tau(\kappa) - \epsilon, \mu_\tau) < \eta.$$

  That is, a shock that reduces relationship strengths arbitrarily little (ϵ) from their equilibrium levels leads to a reliability very close to 0 (within η).
- We say there is *equilibrium robustness* at κ if there is not equilibrium fragility.

Note that the definition of equilibrium fragility makes sense only if $x^*_\tau(\kappa) > 0$ for all large enough τ, and this implies reliability bounded away from zero before the shock, since otherwise there would be no incentive for investment. In the definition of fragility, while shocking $\underline{x}$, we hold investment decisions and entry choices fixed. This corresponds to the assumption, discussed in Subsection 5.2 below, that investments in supply relationships and entry decisions are made over a sufficiently long time frame that firms cannot change the quality of their supply relationships or their entry decisions in response to a shock.

Proposition 3. Under the conditions of Theorem 1, there is equilibrium fragility at any $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ and equilibrium robustness at any $\kappa > \overline{\kappa}$.

Proposition 3 follows immediately from the definition of equilibrium fragility and our previous characterization.

5.2. **Some comments on interpretation.**

5.2.1. *Sources of shocks.* There are many ways in which the small systemic shock involved in our notion of fragility might arise in practice. In the introduction, we discussed the example of a small shock to credit markets making many supply relationships slightly more likely to fail, if they happen to require financing to deal with a disruption. Another example comes from considering new trade regulations: if there is uncertainty about which relationships will be affected by new compliance issues, each supply relationship will be ex ante more prone to disruption. For a third example, consider an increasing backlog in commercial courts—a circumstance that makes contracts more costly to enforce. This again decreases the probability that contracts function in some states of the world—an uncertainty that can affect many players in the supply chain. A final example comes from supply chain stresses due to macroeconomic shocks. Technological factors, such as social distancing measures imposed during a pandemic, can cause stochastic delays and disruptions throughout the dependency links in a supply network. Congestion provides another type of stress. For example, a demand shock to some goods can require ports and transportation hubs to accommodate more traffic, causing queues. These queues delay shipments of goods in many supply networks, even those not affected by the demand shocks that caused the queues (McLain et al., 2021).

5.2.2. *Time scales.* In interpreting our model, a crucial issue is the time scales on which the contagion occurs, and on which decisions are made. We define the short run to be period of time over which firms are *not* able to adjust their relationship strengths. It is this time scale on which the shock we study has its most direct implications in terms of freezing supply chains. Evidence from Barrot and Sauvagnat (2016) suggests that in practice this time scale is on the order of magnitude of several quarters. In contrast, the medium run as a period of time over which adjustments in relationship strength are possible.[26] Thus, following a shock to relationship strengths, in the short run a collapse in production will occur when the supply network is in the fragile regime—that is, holding investments in relationship strengths at their old values. In the medium run, it is possible that firms can choose investment levels that allow the supply network to resume being productive. Our equilibrium selection here is optimistic: it focuses on the outcome where, in recovering from the shock, firms coordinate on the most productive investment equilibrium, and thus limit the losses. Alternatively, once production has become disrupted, it could be that firms stop investing altogether. If such coordination problems occur, disruptions can be longer-lasting and consistent with long-lasting productivity damage following a shock.

6. Discussion of the model

This section discusses our modeling choices. First, in Section 6.1 we highlight which features of our environment are essential, and motivate them. Beyond the key assumptions, we make a variety of assumptions for tractability or simplicity. In Section 6.2, we discuss the robustness of our model to various relaxations of these non-essential assumptions.

6.1. **Essential features.** The three essential features that we highlight are that supply relationships are specific (firm-to-firm) and subject to disruption; that firms endogenously invest to strengthen their relationships; and that production is complex in that it relies on multiple essential inputs at each of a sufficiently large number of layers.

Specific sourcing relationships that are subject to idiosyncratic disruptions are a crucial feature of our model. Supplier relationships have been found to play important roles in many settings—for relationship lending between banks and firms see Petersen and Rajan (1994, 1995); for traders in Madagascar see Fafchamps and Minten (1999); for the New York apparel market see Uzzi (1997); for food supply chains see Murdoch, Marsden, and Banks (2000); for the diamond industry see Bernstein (1992); for Japanese electronics manufacturers see Nishiguchi (1994)—and so on. Indeed, even in fish markets, a setting where we might expect relationships to play a minor role, they seem to be important (Kirman and Vriend, 2000; Graddy, 2006). The importance of specific sourcing relationships in supply networks is also a major concern of the management literature on supply chains (Datta, 2017). Barrot and Sauvagnat (2016) find that firms have difficulty switching suppliers even when they need to do so. These literatures examine many reasons behind firms' reliance on a small number of suppliers to source a given type of input. For example, technological compatibility and geography can limit the pool of potential suppliers; hold-up problems can make trust important; and frequent repeated interactions can help firms to avoid misunderstandings.

[26] Other technological dimensions, such as the complexity m of production, might change on a longer time scale still; see Section 8.4.

The specific relationships that firms maintain for sourcing facilitate the contagion of disruption. Indeed, cascading disruptions (see, for example, Carvalho et al. (2020)) are evidence of firms' reliance on failure-prone sources of supply that cannot be quickly replaced. Interesting qualitative descriptions of cascades of disruption due to idiosyncratic shocks can be found in the business literature. A fire at a Philips Semiconductor plant in March 2000 halted production, preventing Ericsson from sourcing critical inputs, causing its production to also stop (*The Economist*, 2006). Ericsson is estimated to have lost hundreds of millions of dollars in sales as a result, and it subsequently exited the mobile phone business (Norrman and Jansson, 2004). In another example, two strikes at General Motors parts plants in 1998 led 100 other parts plants, and then 26 assembly plants, to shut down, reducing GM's earnings by $2.83 billion (Snyder et al., 2016). Though these particular examples are particularly well-documented, disruptions are a more frequent occurrence than might be expected. In a survey of studies on this subject in operations and management, Snyder et al. (2016) write, "It is tempting to think of supply chain disruptions as rare events. However, although a given type of disruption (earthquake, fire, strike) may occur very infrequently, the large number of possible disruption causes, coupled with the vast scale of modern supply chains, makes the likelihood that some disruption will strike a given supply chain in a given year quite high." An industry study recorded 1,069 supply chain disruption events globally during a six-month period in 2018 (Supply Chain Quarterly, 2018).

Given the frequency of disruptions and the impact these can have on firms' profitability,[27] it is natural that firms take actions to mitigate them. In practice, these investments are often "soft" in nature. A literature in sociology helps document them—a prominent contribution being Uzzi (1997), who offers a detailed account of the systematic efforts and investments made by New York garment manufacturers and their suppliers to maintain good relations. These investments include practices such as building a better understanding of a supplier's or customer's capabilities by visiting their facilities, querying odd instructions to help catch mistakes, building social relationships that span the organizations, and reciprocal gift-giving. Such investments are hard to observe and even harder to verify. This makes them hard to contract on and fits the way we model investments—as a decentralized game of private investment in one's local relationships.[28]

It is crucial for our theoretical results that production is complex. One aspect of complexity is in the horizontal dimension of our network diagrams: firms need multiple inputs via specific sourcing relationships. Barrot and Sauvagnat (2016) provide evidence that supports this assumption. They show that if a supplier is hit by a natural disaster it severely disrupts the production of their customers *and* also negatively impacts their customers' other suppliers. If production were not complex in this way, then these other suppliers would be providing substitute inputs and hence would benefit from the disruption to a competitor rather than being adversely affected. Similarly, this suggests that inventories are not held at high enough levels to avoid disruption, though they

[27]Hendricks and Singhal (2003, 2005a,b) examine hundreds of supply chain problems reported in the business press. Even minor disruptions are associated with significant and long-lasting declines in sales growth and stock returns.

[28]The key feature here is the modeling of investment as a non-cooperative game. Even if firms were allowed to invest in others' relationships they would typically not want to in equilibrium. Following the logic of Bergstrom, Blume, and Varian (1986) each firm will be willing to invest in each relationship (between any two firms) up to the point that the marginal return of the investment equals the marginal benefits. However, the marginal benefits are heterogeneous and it will typically only be possible for this condition to be satisfied by the firm that has the highest marginal benefit of investing.

may help delay it. Another aspect of complexity is that production takes place in multiple layers—i.e., is sufficiently deep. As we show in Section SA4 of the Supplementary Appendix, even without taking any limits, realistic degrees of complexity either in the horizontal or vertical dimension generate patterns that have the key qualitative features driving our main results.

An example of a firm with a relatively complex supply chain is Toyota. After the Great East Japan Earthquake Toyota suffered considerable disruption to its supply network due, in part, to the impact on firms several layers upstream of it. Following this, Toyota made an effort to map out its supply network so it could better anticipate and respond to such problems. It has been reported that despite remaining incomplete this mapping got as far as the tenth level and identified 400,000 items that Toyota sources directly or indirectly (McLain, 2021).

One final observation is that our model can be reinterpreted as describing processes or inputs, rather than firms, as long as the key conditions outlined in this section hold. From this perspective, the phenomena we discuss may happen within a firm or other organization. Increasing the strength of relationships would then correspond to increasing various forms of redundancy or robustness in sourcing. When agency frictions are significant and decisions about robustness are made in a decentralized way, our model of endogenous investments would also be relevant.

6.1.1. *Contrasts with models without the essential features.* In this subsection, we briefly consider two alternative models, which we consider useful benchmarks, to highlight the necessity of key features discussed in the previous section.

Contrast I: Sourcing through spot markets. In this benchmark, each firm sources all its inputs through spot markets, rather than requiring pre-established relationships. The market is populated by those potential suppliers that are able to successfully produce the required input. To keep the technology of sourcing comparable, we posit that, after a buyer places an order in this spot market, there is still a chance that sourcing fails. (A shipment might be lost or defective, or a misunderstanding could lead the wrong part to be supplied.) In other words, we now assume each firm extends relationships *only* to functional suppliers (as opposed to suppliers whose functionality is random, in our main model), but we still keep the randomness in whether the sourcing relationships work.

We now work out which firms can produce, focusing on the example where each firm requires two inputs ($m = 2$). Each firm if multisources by contracting with two potential suppliers of each input ($n = 2$)—selected from the functional ones. Let the probability a given attempt at sourcing an input succeeds be x, independently. The probability that both potential suppliers of a given input type fail to provide the required input is $(1 - x)^2$, and the probability that at least one succeeds is $1 - (1 - x)^2$. As the firm needs access to all its required inputs to be able to produce, and it requires two different input types, the probability the firm is able to produce is $(1 - (1 - x)^2)^2$. In Figure 6 we plot how the probability that a given firm is able to produce varies with the probability its individual sourcing attempts are successful. This probability increases smoothly as x increases. This benchmark thus shows that perfect spot markets remove the discontinuities in our main analysis.

On the other hand, the fragility of complex production remains a concern even in the presence of imperfect substitutes for specifically sourced inputs. More precisely, suppose the quality of a

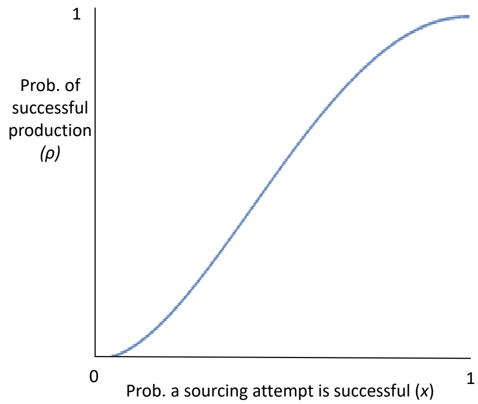


FIGURE 6. The probability of successful production for a firm as market-based sourcing attempts become more likely to succeed.

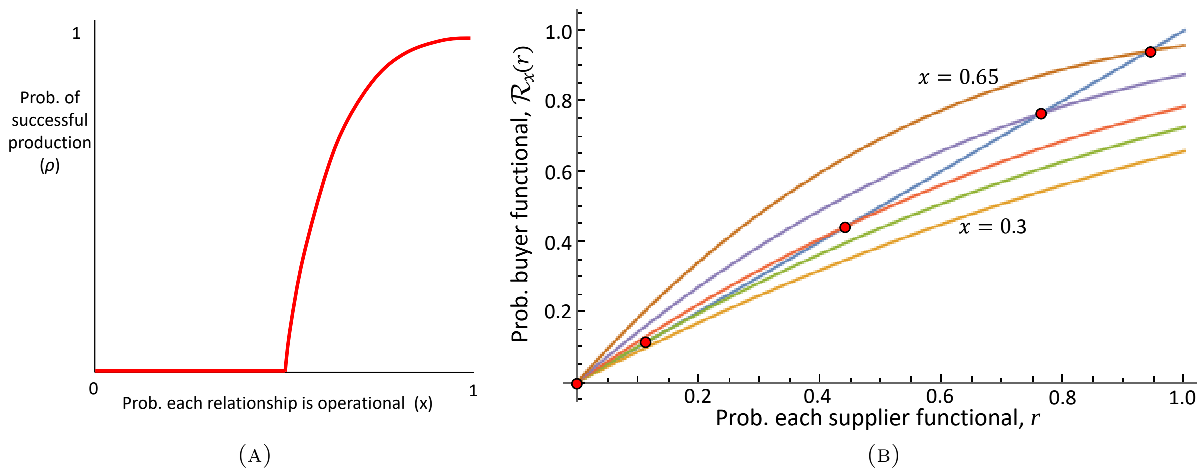


FIGURE 7. (A) The probability of successful production of a simple good ($m = 1$ distinct inputs needed) as relationship strength varies. (B) The probability, $\mathcal{R}_x(r)$, that a focal firm is functional as a function of r, the probability that a random supplier is functional. This plot is for $n = 3$ with $x \in \{0.3, 0.35, 0.4, 0.5, 0.65\}$. The plot parallels Figure 4(A), which depicts the same function for the complex production ($m > 1$) case. The marked intersections with the 45 degree line reflect limit reliability outcomes.

firm's output is discounted by a certain factor whenever it can't source high-quality inputs for all customized inputs required; such a firm instead produces a lower-quality version of its good. Such a shock propagates downstream just as in our main analysis, but where affected firms can still produce low-quality goods.

Contrast II: Sourcing for simple production. To emphasize that it is essential that multiple inputs are sourced through relationships, we consider a benchmark model where each firm requires only a single relationship-sourced input ($m = 1$). Because each firm requires only *one* type of risky input relationship to work, we call such production *simple*.[29] We plot how the probability of

[29]As a matter of interpretation, there may be more than one physical input at each stage. The key assumption is that all but one are sourced as commodities rather than through relationships, and so are not subject to disruption via shocks to these relationships.

successful production varies with relationship strength in Figure 7(a) for an example with $n = 2$. In comparison to the case of complex production illustrated in Figure 3(b), there is a stark difference. For values of $x < 0.5$ the probability of successful production is 0, and for values of $x > 0.5$ the probability of successful production is strictly positive. The contrast is in the nature of the change at $x = 0.5$, where productivity is continuous in the simple-production case, though the derivative is discontinuous.

The intuition is familiar from the networks literature and in particular from studies of contagion (see, for example, Elliott, Golub, and Jackson (2014) in the context of financial contagion, or Sadler (2020) for information diffusion). In the large-depth limit, production will be successful if the supply tree of functional producers upstream of a given firm continues indefinitely, rather than being extinguished due to an excess of failures. This depends on whether the rate at which new branches in the network are created is higher or lower than the rate at which existing branches die out due to failure. It turns out that when $x > 0.5$, a supply tree grows without bound in expectation, while when $x < 0.5$ it dies out.[30] The kink in the probability of successful production around the threshold of 0.5 is related to the emergence of a giant component in an Erdös–Rényi random graph. That continuous phase transition is different from the discontinuous one driving our main results. Indeed, to see the importance of the difference in these phase transitions for equilibrium fragility, consider a productive equilibrium when production is simple. It can immediately be seen from Figure 7(a) that a small shock to reliability x will only cause a small change in the probability of successful production. In contrast to the complex production case, the reliability curve does not become arbitrarily steep as we increase the depth of the supply network.

6.2. **Robustness: Extensions relaxing simplifying assumptions.** In this section we discuss directions in which the analysis can be extended to show that certain simplifying assumptions are not essential to the main findings. We begin with extensions to the model in which firms remain homogeneous (in the distribution of their network positions), and then turn to heterogeneities in the next subsection.

6.2.1. *Investments on the extensive margin.* Our modeling of investments in supply relationships is compatible with multiple interpretations. The first interpretation is that the set of possible suppliers is fixed, and the investment works on the intensive margin to improve the quality of these relationships (e.g., by reducing misunderstandings and so on). The second interpretation is that the investment works on the extensive margin—i.e., firms work to find a supplier willing and able to supply a given required input type, but their success is stochastic. In this interpretation there is a fixed set of n potential suppliers capable of supplying the required input to be found, and each one of them is found independently with probability x_{if}.[31] Conditional on a supplier

[30] Given that each producer has two potential suppliers for the input, and each of these branches is operational with probability x, the expected number of successful relationships a given firm in this supply network has is $2x$. When $x < 1/2$, each firm links to on average less than 1 supplier, and so the rate at which branches in the supply tree fail is faster than the rate at which new branches are created. The probability that a path in the supply network reaches beyond a given tier l then goes to 0 as l gets large and production fails with probability 1. On the other hand, when $x > 1/2$, the average number of suppliers each firm has an operating relationship with is greater than 1 and so new branches appear in the supply tree at a faster rate than they die out, leading production to be successful with strictly positive probability.

[31] This search technology is similar to ball-and-urn models of search, and is compatible with a matching function exhibiting constant returns to scale (see, for example, Hall (1979)).

being found, the relationship is operational. In Supplementary Appendix SA6 we discuss a richer extensive-margin interpretation, and also one that permits separate efforts to be directed to the extensive and intensive margins simultaneously.

6.2.2. *Anticipated shocks.* We assume that the small shock to x is unanticipated. It is straightforward to use our analysis to see that this is not essential. Suppose now all firms anticipate that a shock to x will happen with some probability p_{shock}, and suppose κ is in an open subset of the critical productivities, so that a collapse would have occurred in the baseline model. When we write the profit function for firms in the extended model, there will now be an expectation over the shock's arrival, and this will change profits and the best-response correspondence. However, for a small enough probability of the shock, this change will be small, and the best-response curve in a plot such as Figure 5(c) will move only slightly. Thus the equilibrium will still be on the precipice if it was before. Indeed, for κ not too close to the boundary of the critical productivities, the probability of the shock can be quite substantial without changing the main prediction. The key intuition is that because of externalities, while all firms may prefer a commitment to invest more to avoid being on the precipice (and avoid disruptions due to the shock), the free-riding problem can be too severe to achieve this. Indeed, note that in Figure 5, even fairly large shifts in the best-response curve can leave the equilibrium on the precipice, implying that these free-riding effects are considerable.

6.2.3. *Shocks to firms.* The fundamental source of shocks in our model is at the level of links in the supply network. One could also consider shocks directly hitting firms—i.e., shocks to nodes in our supply network rather than links. The key conclusions regarding precipices are robust to adding this source of shocks, or even making node-level shocks the main source of disruptions.[32] We focus on shocks to relationships for simplicity, but the mathematical forces we identify also operate in alternative models of disruption.

6.2.4. *Endogenous entry.* Our model takes the varieties in the market as given, and does not include entry decisions by firms. This is done to keep the model simple. However, the key insights carry over in a model with entry.

Consider an extended model where there is an entry stage preceding all others. At this stage, a firm (i, f) pays a sunk entry cost $\Phi(f)$, where Φ is a strictly increasing function. Then the measure of varieties in each product is set equal to the measure of entering firms (i.e., each entering firm produces one variety), and the "investment game" where firms choose their relationship strengths proceeds as we described earlier. A (nontrivial) symmetric equilibrium is now one where $\overline{f} > 0$ firms enter in each product and the expected post-entry profits of the marginal firm are equal to its entry cost.

The main observation is that this model also features an open set of productivity parameters κ where equilibrium production is on the precipice. The basic logic is as follows. When the supply network is reliable and gross profits after entry are high, firms want to enter. As they enter, competition drives down gross profits and makes it less appealing to pay costs to make relationships strong. Recalling Figure 5, the key to the argument is that increasing entry moves the best-response curve leftward for a given κ. So, in equilibrium, relationships get weaker as more

[32]This variant of the model is worked out in Elliott and Golub (2021).

entry occurs. The question is where this dynamic stops. The precipice is a natural stopping point. Once the investment level is x_{crit}, reliability can adjust down until further entry is deterred, while investment remains the same. By reasoning parallel to that of our main results, this state is reached for an open set of model parameters.

6.3. **Robustness to heterogeneity.** We have so far assumed for tractability that all firms have identical network positions ex ante. One might suspect that the regularity of the network structure, or some other kind of homogeneity, plays an important role in generating the sharp transition in the probability of successful production. In contrast with the key roles played by complexity, specific sourcing, and depth, homogeneities across the firms are not important to our main points. To establish this, we now discuss extensions with two forms of heterogeneity in firm type.

6.3.1. *Partial knowledge of depth.* We have assumed that firms make investments ex ante of the realization of their supply network depths, so that when they face their investment decisions, they have identical decision problems. This implies that they all have identical investment incentives, a symmetry that simplifies the analysis considerably. However, in practice, firms do have some information about the depth of their supply networks. We now introduce this into the model and give conditions under which our main findings extend.

We refer to the model analyzed in Sections 3–5 as the *baseline model*, and we will define the model with partial knowledge of depths with reference to it.

The firms are now partitioned into two types: low-depth firms, with depth less than or equal to $\overline{d}_l \geq 0$, constitute a proportion $\mu_l < 1$ of the population, and high-depth firms, with depth greater than $\overline{d}_l$, constitute the remainder. (We discuss the case of more than two types below.) Firms' information about depth consists of knowing which type they are. We use the subscript l for low-depth firms and h for the high-depth firms.

Each type of firm has gross profits from production equal to $g_t(r_t)$, where r_t is the aggregate reliability of that type of firm ($t = l$ for low-depth firms, and $t = h$ for high-depth ones). This amounts to assuming that each type of firm sells into a separate market, and its gross profits are dependent on competition in that market through r_t. Similarly, each type of firm has a cost function $\frac{1}{\kappa_t} c_t$. The model is otherwise the same, with the functions g_t and c_t satisfying the assumptions made on g and c, respectively, in the homogeneous model.

We now extend the definition of equilibrium to allow low-depth and high-depth firms to behave differently; equilibrium again refers to a symmetric Nash equilibrium that is not Pareto-dominated by any other symmetric equilibrium, but where symmetric now means that strategy depends only on type. We let the vector $\boldsymbol{x}^*_\tau(\kappa_l, \kappa_h)$ denote relationship strengths for the low-depth and high-depth firms in such an equilibrium.

Our focus will be on the high-depth firms. We thus define an equilibrium to be fragile if, for large enough typical depth, a very small shock causes the *high-depth firms* to have very low reliability.

Definition 3 (Equilibrium fragility with partial knowledge of depth)**.** There is *equilibrium fragility* at $\boldsymbol{\kappa} = (\kappa_l, \kappa_h)$ if for any $\eta, \epsilon > 0$, for large enough τ, we have

$$\rho_h(\boldsymbol{x}^*_\tau(\boldsymbol{\kappa}) - \epsilon, \mu_\tau) < \eta.$$

That is, a shock that reduces relationship strengths arbitrarily little (ϵ) from their equilibrium levels leads to a reliability very close to 0 (within η) for the high-depth firms.

Proposition 4. Suppose there is equilibrium fragility at κ in the baseline model under gross profit function g. Then in a partial knowledge of depth model with $g_h = g$, the following holds: If $x^*_h(\kappa) > 0$ for all sufficiently large τ (i.e. there is positive investment), then there is also equilibrium fragility.

To gain some intuition for the result, note that in a (symmetric undominated) equilibrium, the high-depth firms take the reliability of the low-depth firms as given. In particular, all low-depth firms that the high-depth firms source from are at depth exactly $\overline{d}_l$ and hence have the same reliability. For $\kappa_h = \kappa$ the high-depth firms are then in an equivalent position to firms in our baseline model, except that the firms at depth $\overline{d}_l$ play the role of the depth-zero firms in the baseline model. In that model, the depth zero firms have reliability 1. In the partial depth knowledge model, the depth $\overline{d}_l$ firms have some reliability $\overline{r} \leq 1$. Nevertheless, as long as $\overline{r} \geq \overline{r}_{\text{crit}}$ this difference does not matter for τ sufficiently high. The reason for this is that in both cases, given the same investment decisions for the high-depth firms, limit reliability will be the same as in the baseline model for those firms by the argument in Subsection 3.1.1. Our assumption that there is equilibrium fragility at κ in the baseline model implies that, in that model, the limit of equilibrium investment is x_{crit}. The proof shows that there will also be limit investment x_{crit} for the high-depth firms in the symmetric undominated equilibrium of the extended model (as $\tau \to \infty$).

This leaves the case when the reliability of the depth $\overline{d}_\ell$ firms is $\overline{r} < \overline{r}_{\text{crit}}$. We just need to show that in this case limit reliability is either 0 or $\overline{r}_{\text{crit}}$. Hence it is sufficient to rule out limit equilibrium investment $x > x_{\text{crit}}$. Such investment cannot be supported in equilibrium for the same reasons that it could not be supported in the corresponding baseline model.

It is important for interpreting Proposition 4 that we know the high-depth firms are sometimes in the first positive-investment case, since positive investment is required for equilibrium fragility to make sense. To show that the positive-investment assumption is not vacuous, note that we can always set κ_l sufficiently high to induce equilibrium investments for the low-depth firms above x_{crit}, which is sufficient for ensuring the reliability of depth $\overline{d}_l$ firms is above r_{crit}.

When there is equilibrium fragility, it is only the high-depth firms that are affected. Moreover, the low-depth firms operate independently of the high-depth firms, and so even if production is disrupted for all the high-depth firms, the low-depth firms can still produce. This readily extends to a finer partitioning of firms by their depth. In this case, those firms in all but the highest depth group have bounded depth and behave like the low-depth firms; the highest-depth group would behave like the high-depth firms. Extending the model to include partial knowledge of depth, as we have done in this section, changes the quantitative implications of the model—fewer firms in a supply chain will be on the precipice—but not the qualitative implications insofar as a positive mass of the firms will be on the precipice. The discussion in Section 6.1 suggests that a considerable number of high-value products would be included in this set.

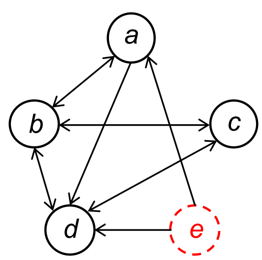
It is worth noting that, for simplicity, we have firms depend on others upstream of themselves (their suppliers, their suppliers' suppliers, and so on) but not downstream.[33] If the profits of low-depth firms were instead derived from selling to high-depth firms, then the inability of the high-depth firms to produce would, via reduced demand for the low-depth firms' products, affect

[33]In our microfoundations (Appendix C), the upstream but not downstream dependence is modeled by letting all firms sell to final consumers, and letting all profits be derived from such sales.

the profitability of production for the low-depth firms and they might not be able to profitably produce, either. This type of propagation could be incorporated in to the specification of the gross profit function g. Baqaee (2018) offers a detailed study of such contagion of failure in both directions in a different model.

6.3.2. *Heterogeneous product networks.* We now consider an extension of our model with flexible heterogeneity in network structure. We focus in this subsection on the mechanics of how reliability depends on relationship strengths. In the Supplementary Appendix (Section SA5) we flesh out the details rigorously and also present an extension of the endogenous investment model.

The key point in this heterogeneous extension is that the number of inputs required, m, and the number of potential suppliers, n, is no longer held fixed across the supply network; these are allowed to vary with the product. More formally, there is a finite set of products, $\mathcal{I}$. Each product $i \in \mathcal{I}$ is associated with a *product complexity* m_i and a finite set of inputs $I(i) \subseteq \mathcal{I}$ of cardinality m_i. Thus, the number of inputs required can be different for different products. The neighborhoods $I(i)$ define the *product dependence graph*, with the convention that arrows are directed upstream. For each firm producing product i, each needed input j can be supplied to the firm by n_{ij} potential suppliers. For each pair $i, j \in \mathcal{I}$, there is a relationship strength x_{ij} such that every link from a firm producing i to a potential supplier of product j has a probability x_{ij} of being operational, irrespective of depth. The product dependence graph is illustrated in Figure 8(A), while the heterogeneities in the number of multisourcing possibilities and in link strength are illustrated in panel (B). We denote by $\boldsymbol{x} = (x_i)_{i\in\mathcal{I}}$ the profile of relationship strength vectors for the different products. We let $x_i = (x_{i,j})_{j\in I(i)}$.



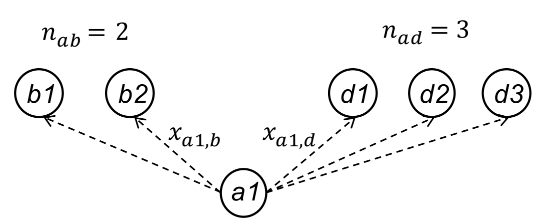


FIGURE 8. Panel (A) depicts a product dependence graph in the heterogeneous case. An arrow from a to b means that b is an input required in the production of a, i.e. $b \in I(a)$. Product b requires $m_b = 3$ inputs (products a, c and d). Other products require 2 inputs and thus $m_i = 2$ for $i \in \{a, c, d, e\}$. There are two strongly connected components: $\{a, b, c, d\}$ and $\{e\}$ (which is highlighted). Panel (B) shows a potential supply network for variety $a1$. Here input products b and d are required, as indicated in (A), and the number of potential suppliers of each input can be different: there are $n_{ab} = 2$ potential suppliers of input b while there are $n_{ad} = 3$ potential suppliers of input d. Moreover, the relationship strength is input-specific, with $x_{a1,b}$ being possibly different from $x_{a1,d}$.

Analogously to our main model, we introduce depths of each variety. Depth-zero varieties are those that require no specifically sourced inputs. We specify the matching process so that each variety sources from varieties of strictly smaller depth. Subject to this, we allow a flexible distribution of input depths, relaxing the requirement from the baseline model that all suppliers of a given firm are exactly one level less deep. We are again interested in the limit as depths of most firms become arbitrarily large. In Appendix SA5 we present the details of this setup and characterize this limit, paralleling Section 3.1 in the homogeneous case. This analysis yields a *limit reliability correspondence* $\boldsymbol{\rho}$ mapping $\boldsymbol{x}$ to a profile $(r_i)_{i\in\mathcal{I}}$ of reliabilities. The correspondence has the property that, for any $\boldsymbol{x}$, if depths are sufficiently large, the reliabilities are arbitrarily close to those given by $\boldsymbol{\rho}$.

Our first main result is that the sharp transition in the production function persists in the heterogeneous model. There are values of the vector $\boldsymbol{x}$ where arbitrarily small shocks to relationship strengths result in large drops in reliability. The following definition is helpful for making this point:

Definition 4 (Critical product)**.** The product i is *critical at* $\boldsymbol{x}$ if the product has positive reliability $\rho_i(\boldsymbol{x}) > 0$ at strengths $\boldsymbol{x}$, but has reliability $\rho_i(\boldsymbol{x}') = 0$ at any profile $\boldsymbol{x}'$ of strengths that is elementwise strictly lower than $\boldsymbol{x}$.

In Section SA5.1 we generalize Proposition 1 by giving conditions guaranteeing that *critical products exist* in the heterogeneous model. The key condition is, as before, that production requires $m_i \geq 2$ distinct inputs at each step.

We next show that supply networks in the presence of heterogeneities feature a *weakest link* property (Proposition 5): When one product is critical, all products that rely directly or indirectly on it are critical as well.[34] This result uses the notion of a product dependence graph, whose nodes are a set $\mathcal{I}$ of products, and products have directed links to the input products they require (see Figure 8(A) for an illustration).

Proposition 5 (Weakest link property)**.**

(i) Suppose product i is critical. Then any product that has a directed path to i in the product dependence graph is critical.

(ii) Let $\mathcal{I}^{SC} \subset \mathcal{I}$ be a set of products that are part of a strongly connected component of the product dependence graph. Then any equilibrium with positive effort is such that either all producers of products $i \in \mathcal{I}^{SC}$ are critical or no producers of products $i \in \mathcal{I}^{SC}$ are critical.

Proposition 5 asserts that supply networks suffer from a weakest link phenomenon. First, in Part (i), we see that if a product is critical, then any small systemic shock (which causes it to fail, by definition of criticality) causes the failure of all other products that use it as an input, directly or indirectly. For example, in Figure 8(A), we see that if product b is critical, then a small systemic shock to relationship strengths causes its production to fail and will also cause products a, c, d and e to fail. Products a, c, d fail because they use b directly as an input. Product e fails because it indirectly uses product b as an input (through a and d).

Second, in Part (ii), we begin by recalling that a *strongly connected component* of the product dependence graph is a set of nodes (i.e. products) where each node can be reached from any other node in that set by following some directed path of dependencies. For example, in the product dependence graph of Figure 8(A), the nodes represented by full black circles (a, b, c and d) form such a strongly connected component. Node e is not part of this component since it cannot be reached from the other nodes by following the direction of the arrows. A strongly connected component of products is a set of products such that each product is used as input (directly or

[34]Thus, as in Bimpikis, Ehsani, and Ilkılıç (2019b) the social planner has different gains from intervening in different parts of the network (see their Proposition 8).

indirectly) by every other product of that set. This is precisely the intuition behind the fact that such a component is only as strong as its weakest link. Indeed, if one product is critical, then all products in that component are also critical at the same time.

In Section SA5.3 of the Supplementary Appendix, we provide proofs supporting this section and illustrate both the discontinuities and the weakest link property with numerical examples. We also extend our endogenous investment model and show that the configurations we have described are consistent with strategic investment in relationship strength.

7. Related literature

We have already discussed many papers that are relevant for motivating our assumptions or interpreting our results in Section 6. In this section we review several high-level connections to related literatures not covered by our previous discussions.

There has been considerable recent interest in markets with non-anonymous trade mediated by relationships.[35] The work most closely related to ours in this area also studies network formation in the presence of shocks. This includes work in the context of production (e.g., Fafchamps (2002), Levine (2012), Brummitt, Huremović, Pin, Bonds, and Vega-Redondo (2017), Bimpikis, Candogan, and Ehsani (2019a), Yang, Scoglio, and Gruenbacher (2019), Amelkin and Vohra (2019)), work on financial networks (e.g., Cabrales, Gottardi, and Vega-Redondo (2017), Elliott, Hazell, and Georg (2018), Erol (2018), Erol and Vohra (2018), Jackson and Pernoud (2019)), and other contexts (e.g., Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011) Jackson, Rodriguez-Barraquer, and Tan (2012), Talamàs and Vohra (2020)). More broadly, the aggregate implications of relying on relationships to transact have been studied across a variety of settings. For work on thin financial markets see, for example, Rostek and Weretka (2015), for buyer-seller networks see, e.g., Kranton and Minehart (2001), and for intermediation see, e.g., Gale and Kariv (2009). Our work focuses on network formation for *production* and emphasizes the distinctive network formation concerns that arise due to strong complementarities. At a methodological level, we offer an approach that may be useful more broadly. Agents make a continuous choice that determines the probability of their relationships operating successfully. The links that form may, however, fail in a "discrete" (i.e., non-marginal) way. The first feature makes the model tractable, while the second one yields discontinuities in the aggregate production function and distinguishes the predictions from models where the aggregate production function is differentiable. It might be thought that aggregating over many supply chains, these discontinuities would be smoothed out at the level of the macroeconomy; we show they are not.

There is a vibrant literature in macroeconomics on production networks. This literature dates back to investigations of the input-output structure of economies and the implications of this (Leontief, 1936). Carvalho and Tahbaz-Salehi (2019) provides a comprehensive survey.

Two recent developments in the literature are particularly relevant to our work: (i) the modeling of the endogenous determination of the input-output structure; and (ii) a firm-level approach as opposed to considering inter-industry linkages at a more aggregated level. Some of the most relevant work on these issues includes Atalay, Hortacsu, Roberts, and Syverson (2011), Oberfield (2018), Carvalho and Voigtländer (2014), Acemoglu and Azar (2020), Boehm and Oberfield (2020),

[35] A literature in sociology emphasizes the importance of business relationships, see for example Granovetter (1973) and Granovetter (1985). For a survey of related work in economics see Goyal (2017).

Tintelnot, Kikkawa, Mogstad, and Dhyne (2018), Liu (2019), Acemoglu and Tahbaz-Salehi (2020), and König, Levchenko, Rogers, and Zilibotti (2019). Baqaee and Farhi (2019) and Baqaee and Farhi (2020) focus specifically on the implications of nonlinearities, and discuss how nonlinearities in firms' production functions propagate and aggregate up. Whereas they focus on smooth nonlinearities, we show that especially extreme nonlinearities—discontinuities—naturally come from complex supply networks. We have discussed throughout how our explicit modeling of sourcing failures in complex production at the micro level gives rise to new effects (see Section 6.1.1).

The strong complementarities in production in our model are crucial for creating fragility, and this aspect of our work builds on a large literature following the seminal work of Kremer (1993). This literature argues that complementarities can help provide a unified account of many economic phenomena. These include very large cross-country differences in production technology and aggregate productivity; rapid output increases during periods of industrialization; and the structure of production networks and international trade flows; see, among many others, Ciccone (2002), Acemoglu, Antràs, and Helpman (2007), Levchenko (2007), Jones (2011) and Levine (2012).[36] A possible concern with this literature is that if firms were allowed to take actions that mitigate supply risks (for example, by multisourcing), that this would endogenously dampen the crucial complementarities. Our work helps address this concern. We show that even when firms can take actions to mitigate their risk by multisourcing, a very severe form of equilibrium fragility—and hence production complementarities—arises.

At a technical level, our work is related to a recent applied mathematics literature on so-called *multilayer networks* and their phase transitions (Buldyrev, Parshani, Paul, Stanley, and Havlin, 2010). Discontinuities such as the one we study are termed first-order phase transitions in this literature.[37] Buldyrev, Parshani, Paul, Stanley, and Havlin (2010), and subsequent papers in this area such as Tang, Jing, He, and Stanley (2016) and Yang, Scoglio, and Gruenbacher (2019), study quite different network processes, typically with exogenous networks. We show that discontinuous phase transitions arise in canonical models of production networks, once specific sourcing relationships are taken into account. Importantly, we also endogenize investments affecting the probabilities of disruption (which are taken to be exogenous in this literature) and elucidate a new economic force endogenously putting equilibria on a precipice. Dasaratha (2021) studies endogenous investment in a random network in an information-sharing model without aggregate uncertainty. Predating the recent literature just discussed, Scheinkman and Woodford (1994) used insights from physics models on self-organized criticality to provide a "sandpile" model of the macroeconomy in which idiosyncratic shocks have large aggregate effects.[38] The setup and behavior of the model are rather different from ours: the main point of commonality is in the concern with endogenous fragility. In our model, investments in supply relationships leave the supply network robust to idiosyncratic shocks, but very sensitive to arbitrarily small systemic shocks to relationship strength.

[36]Prior to his literature, Jovanovic (1987) examines how strategic interdependencies or complementarities can produce aggregate volatility in endogenous variables despite only seemingly "diversifiable" idiosyncratic volatility in exogenous variables.

[37]These can be contrasted with second-order phase transitions such as the emergence of a giant component in a communication network, which have been more familiar in economics—see Jackson (2008).

[38]Endogenously, inventories reach a state analogous to a sandpile with a critical slope, where any additional shock (grain dropped on the sandpile) has a positive probability of leading to an avalanche.

8. Concluding discussions

We conclude by sketching some implications of our analysis. We first note some basic welfare implications of the model, complementing the descriptive analysis of fragility that has been our focus so far. We then turn to the implications for policy approaches aimed at improving the robustness of supply networks. Finally, we briefly suggest some potential applications of the precipice phenomena in macroeconomic modeling and studies of industrial development.

8.1. **Welfare implications.** In Section 4.1 we showed that a planner would never choose investments that result in the supply network being fragile. On the other hand, in Section 5 we found that decentralized investment choices often result in fragile supply networks. We now show that there is systemic underinvestment in reliability at fragile equilibria and discuss the structure of externalities.

There are three externalities present when a firm chooses its reliability. First, there is the non-appropriability of consumer surplus. Output is higher when more firms function, and this is not fully internalized in a firm's reliability choice even with respect to adding *its own* variety to the mix, since the firm appropriates only part of the surplus. Second, there are reliability spillovers: firms don't internalize their contributions to helping *other* varieties function. The reliability of a firm's intermediate good production increases the reliability, and hence profitability, of those firms sourcing from it, the firms sourcing from these firms, and so on. Both the non-appopriability of consumer surplus and the reliability spillover lead to underinvestment in reliability (all else equal). On the other hand, there is also a business-stealing effect. Producers of a good make fewer sales when more producers of the same good are functional. This is a force for overinvestment, all else equal, as firms jockey to capture a larger share of the market.[39]

Consider a critical equilibrium. We know from Section 4.1 that reliability investments are never efficient in a fragile equilibrium. The marginal improvement in reliability from a uniform increase in investments is arbitrarily large—indeed, this is how we concluded that the planner never chooses reliability that leads to a fragile supply network. So, in a critical equilibrium, the non-appropriability of consumer surplus and reliability spillovers together dominate the business-stealing effect, and there is underinvestment in relationships relative to an efficient benchmark.

In fact, something much stronger than this is true: there is *always* underinvestment in relationship strengths in a productive equilibrium. This result follows from the proof of Lemma 5 in the appendix, which underlies our selection of the high-reliability equilibrium as the efficient one.

Corollary 2. Suppose we have a symmetric equilibrium with relationship strengths $x \geq x_{\text{crit}}$. Then[40] $x < x^{SP}(\kappa, \mu)$.

This motivates an interest in policies that increase investment, as this both increases welfare and has the potential to mitigate fragility. We turn to these next.

8.2. **Implications for policies supporting investment and robustness.** Counteracting underinvestment and removing fragility is not straightforward. Here we discuss several intuitive policy

[39] Higher investments in reliability increase the probability a firm is able to function and compete, and so can be interpreted as an investment in (stochastic) entry. In light of this it is unsurprising that the usual non-appropriability of consumer surplus and business-stealing effects are present.

[40] Note that $x^{SP}(\kappa, \mu)$ is single-valued for such values of x, so we may treat x^{SP} as a function here, abusing notation.

interventions that have limited effects, and others that have more potential to qualitatively change the robustness of the system.

As we noted in Section 5.2.2, we view investments as being chosen in the medium run, an ex ante decision that does condition on shock realizations. Some policies will affect investment decisions on this timescale. In contrast, a shock causes disruption in the short run, a timescale on which firms cannot adjust their investment decisions. However, some policies may be able to react to shocks on this shorter timescale. We will analyze policies in both horizons, beginning with the medium run investment decision.

8.2.1. *Marginal interventions on investment incentives.* Suppose investment in link strength is subsidized to reduce the marginal cost of investments, making the cost of a given investment $1-\theta$ times its original cost. More precisely, we make the gross profit equal to

$$\Pi_{if} = \underbrace{P(x_{if}; x)}_{\text{prob. functional}} \quad \underbrace{g(r)}_{\substack{\text{gross profit}\\ \text{if functional}}} \quad - \underbrace{\left(\frac{1-\theta}{\kappa}\right) c(y_{if})}_{\text{cost of effort}}.$$

Suppose the status quo were at a critical equilibrium with reliability $r = \underline{r}$ (see Figure 5(B) for an illustration). A subsidy of this form will shift the red best response curve in Figure 5(B) to the right. At the margin this will increase reliability, but have a very limited effect on equilibrium investments for high values of τ. Thus the equilibrium will remain fragile. The same argument can be used for increasing κ. This leads to the counterintutive conclusions that a policy intervention that works well in terms of increasing reliability in normal times can still fail to remove fragility to aggregate shocks.

Indeed, a similar analysis applies for many other interventions that shift the best response curve to the right—including, for example, improvements in institutions that increase $\underline{x}$ marginally.

8.2.2. *Larger interventions to improve investment.* While marginal interventions typically fail to resolve fragility, non-marginal interventions might work, for the reasons apparent in the earlier discussion (see also Figure 5(B)). They can shift the equilibrium into the non-fragile regime from the fragile one. The kinds of policies that would work here would be substantial subsidies to investments in reliability strength or actions that increase profitability in an industry. This could be done in a variety ways: large tax exemptions might be granted or competition might be limited by, for example, improved patent protections.

Such interventions have the flavor of "big push" policies advocated for developing counties to escape poverty traps, but are different in a few ways from the standard analysis (Murphy et al., 1989).[41] The standard theory underlying a "big push" approach is one of multiple equilibria: there is a low-output equilibrium with traditional production where demand is low and traditional technologies are more cost-effective, and a high-output equilibrium with industrialized production where demand is high and industrial technologies are more cost-effective. For an economy to grow out of the bad equilibrium, it must do so in a balanced way to take advantage of one sector's growth fueling other sectors' demands, beyond external economies of scale that are confined in a given industry (Rosenstein-Rodan, 1943). A temporary intervention that subsidizes industrial technology can facilitate this shift. Crucially, the good equilibrium is sustained by increased production

[41]See also Okuno-Fujiwara (1988) and Rodrik (1996).

under the same institutions. Once a temporary intervention is removed, the technology and other economic primitives remain unchanged: in particular, there is no long-term government support of production, and no institutional changes.

In our analysis we select undominated equilibria and so there is no scope for policies to facilitate coordination on better equilibria. However, without this selection temporary interventions could shift an economy out of a bad equilibrium and into a good one, just as in "big push" theories. Nevertheless, in the fragile regime which is our main focus, all of the productive equilibria involve fragility, and so no temporary policy can remove it. To resolve fragility, permanent intervention on costs or production technologies is necessary.[42] To summarize, the comparison to classic big-push models is interesting: we also find an important role for non-marginal interventions, but for different reasons and with different implications for policy.

8.2.3. *Targeting interventions.* Our analysis offers some tentative suggestions for how interventions should be targeted to best address fragility. For targeting questions to make sense, there must be some heterogeneity across the economy. One type of heterogeneity was introduced in the extension of Section 6.3.1 with partial knowledge of depth. We first consider interventions that will take a supply network out of the fragile regime in the environment presented there. In any productive equilibrium, the equilibrium reliability of the low-depth firms does not affect the limit reliability of the high-depth firms. Hence further improvements in reliability for the low-depth firms will have no impact on the reliability of high-depth firms as their depth gets large. On the other hand, targeting the high-depth firms for non-marginal interventions such as those discussed above may be able to move the supply network out of the fragile regime.

This relates to the issue of reshoring—bringing production that had been outsourced to offshore suppliers back into a rich country. The rationale for reshoring through the lens of the model is as follows. Links involving long-distance shipping to different countries are more prone to disruption and may occur under weaker institutions. Reshoring can therefore be modeled as improving the strength of relationships among those firms involved, or improving the technology for investing in those relationships. The analysis in the previous paragraph suggests that if reshoring targets parts of the supply network that are of low depth (and known to be so by the participating firms), the impact is likely be relatively limited. Reshoring high-depth, complex, products, on the other hand, can be quite effective.

Targeting can also be considered in the model of Section 6.3.2 with richer heterogeneity in the production structure. There we showed that there is generally a "weakest link" phenomenon in which all firms downstream of an industry that is fragile are also fragile (Proposition 5). Thus, policymakers should be especially interested in identifying upstream, or central, complex industries in a critical equilibrium. We would expect the industries at risk to be ones with relatively low profit margins and relatively few sourcing options per firm. Beyond these guidelines, a diagnostic indication of being in or near a critical equilibrium is that reliability is very sensitive to changes in κ (e.g., changes in profitability). Critical industries also have the property that changes in κ are incident only on ρ in equilibrium (as discussed above).

[42] It may be that a more productive economy is capable of supporting institutional reforms or innovation that, for example, change κ. This is a multiple equilibrium story in a larger model, but is still markedly different from the classic big push story as described above.

8.2.4. *Reactive interventions: The short run.* As discussed in Section 5.2.2, the extreme effects of fragility occur in the short run (without giving firms a chance to adjust their investment decisions). Thus, there is a case for quick, reactive short-term government intervention, if such interventions can directly address an aggregate shock. For example, suppose shipping is constrained by regulations that limit the throughput of ports, or short-term shortages of labor in the transport industry. These problems can be modeled as reducing the probability of supply relationships throughout an economy operating normally. Our analysis shows that this can lead to cascading disruptions and production freezes when supply networks are fragile. However, if the government is able to quickly counteract these shocks—e.g., relaxing regulations, or mobilizing military personnel to compensate for the short-term labor shortage—then that can be effective. Moreover, firms' anticipation of such reactions does not significantly change their investment incentives, as long as shocks are fairly rare. (The reasoning here is analogous to that in Section 6.2.2.) Thus, there is no concern of crowding out those incentives. So, perhaps counterintuitively, reactive interventions like this can be more effective in avoiding fragility than more forward-looking policies that seek permanent improvements in reliability (as discussed above).

8.3. **Embedding precipices in a macroeconomic model: Aggregate volatility.** So far we have focused on a single complex supply network with particular parameters, interconnected by specific-sourcing relationships. The larger economy can be thought of as consisting of many such supply networks, each one small relative to the economy. Our question in this section is what the fragility of some of these particular supply networks implies for the reaction of the entire economy to shocks. Our main finding is that while the severe amplification of shocks remains, the conclusion is nuanced by embedding individual supply networks in a richer setting.

Suppose there are many supply networks operating independently of each other, with heterogeneity across supply networks but, for simplicity, homogeneity within each network. The parameters of these different supply networks, including their complexities m and multisourcing numbers n are drawn from a distribution. We know from the above that a small shock to relationship strength can discontinuously reduce the production of some of these supply networks. We now point out that a small shock can have a large macroeconomic effect, and that the structure of the fragile regime is essential for this.

For simplicity, fix the function $c(\cdot)$ to be the same throughout the economy.[43] A given supply network is then described by a tuple $\mathfrak{s} = (m, n, \kappa)$. We consider the space of these networks induced by letting the parameters m, n and κ vary. In particular, we let $\mathcal{M}$ be the set of possible values of m, the set of integers between 1 and M; we let $\mathcal{N}$ be the set of possible values of n, integers between 1 and N, and we allow $\kappa \in \mathcal{K} = [0, K]$. The space of possible supply networks is now $\mathcal{S} = \mathcal{M} \times \mathcal{N} \times \mathcal{K}$. We let Ψ be a distribution over this space, and assume that it has full support.

In some supply networks there will not exist an equilibrium with positive production, which we henceforth call a productive equilibrium (for example, when κ is sufficiently low fixing the other parameters). Consider now those supply networks for which there is a productive equilibrium. There are two possibilities. It may be that the only supply networks for which there is a positive

[43]This could also be drawn from a distribution, but the notation would be more cumbersome.

equilibrium have $m = 1$.[44] That is, the only supply networks with positive reliability are simple. In this case, there is no aggregate fragility.

But if, in contrast, $\mathcal{S}$ contains supply networks where production is *not* simple, then we will have macroeconomic fragility. Indeed, an immediate consequence of Theorem 1 is that if there are some complex ($m \geq 2$) supply networks with positive equilibria, then some of the lower-κ networks with the same (n, m)—which are included in $\mathcal{S}$—are in the fragile regime. The measure that Ψ assigns to supply networks in the fragile regime is positive. Thus, a shock to relationship strengths will cause a discontinuous drop in expected aggregate output. On the other hand, in contrast to the situation with a single supply network, the reduction in output need not be all the way down to zero: it will be a discontinuous loss of some fraction of the output.

So far we have looked at the case in which the different supply networks operate independently and all business-to-business transactions occur through supply relationships confined to their respective supply networks. As we have emphasized, such specific relationships correspond to inputs tailored to the specifications of the business purchasing them. They are not products that can be purchased off-the-shelf. However, many other inputs are sourced in different ways. For example, most business use computers, and buy them off-the-shelf rather than through the specific-sourcing relationships we have focused on. So far we have abstracted from any interdependencies between businesses created by such arm's-length purchases. However, these interdependencies might matter. If a small aggregate shock causes the collapse of some supply networks, the inputs available to other supply networks that managed to remain functional become scarcer and more costly. This effectively damages the productivity of these other supply networks, and when they reoptimize, some of them that were not previously on a precipice will now find themselves there. Thus, they will be sensitive to further aggregate shocks. The key is that being on the precipice is not a fixed attribute of a supply network's structure, but in fact dependent on its productivity. Thus, even with market-mediated spillovers, the productivity damage of collapses leads to domino effects where, iteratively, previously robust parts of the economy become fragile.[45]

8.4. **Some implications for industrial development.** The comparison between the production of complex and simple products developed in Section 6.1.1 has interesting implications for the complexity of technologies used across countries, and for industrial development. While a full analysis of this is beyond the scope of this paper, Figure 9 illustrates the main point. We posit that the revenue of a product is the associated level of production reliability multiplied by the value of the product, which is higher for more complex products. We then plot the two revenue curves, one derived from complex products (Figure 3) and one for simple products (Figure 6.1.1(a)). Small increases in quality of commercial institutions can make more complex production technologies viable and yield discontinuous benefits to an industry by enabling a transition from simple to complex production.

Even this very rudimentary theory of development and industrialization fits a number of stylized facts: (i) Industrialization, when it occurs, is rapid and economic output increases dramatically. (ii) At the same time, the share of the value of total production that can be attributed to intermediate

[44]For example, it might be that $M = 1$ so that only simple production is feasible, or K might be sufficiently low that only simple production has a positive probability of being successful.

[45]In Appendix SA3 we provide a numerical illustration of this phenomenon.

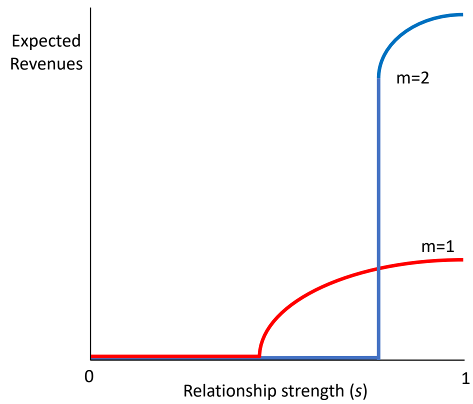


Figure 9. A contrast of the $m = 1$ and $m = 2$ cases with the degree of multisourcing being held at $n = 2$. Expected revenues are on the vertical axis. This is a product of the probability of successful production and price of goods. The case in which the complex good retails for a price of 1 while the simple good retails for a price of 1/4 is illustrated.

inputs increases quickly (Chenery, Robinson, and Syrquin, 1986). (iii) The quality of institutions, and particularly those related to contracting, can help explain what kinds of production different economies can support (Nunn, 2007) and hence cross-country differences in development (Acemoglu and Johnson, 2005), wages and productivities (Jones, 2011). (iv) Finally, more complex supply chains are associated with higher rates of disruption (Craighead, Blackhurst, Rungtusanatham, and Handfield, 2007).[46]

A full study of these issues obviously requires more detailed modeling. Nevertheless the discontinuities we identify may have a useful role to play in theories relating the quality of institutions and an economy's production possibilities.

References

Acemoglu, D., P. Antràs, and E. Helpman (2007): "Contracts and Technology Adoption," *American Economic Review*, 97, 916–943.

Acemoglu, D. and P. D. Azar (2020): "Endogenous Production Networks," *Econometrica*, 88, 33–82.

Acemoglu, D. and S. Johnson (2005): "Unbundling Institutions," *Journal of Political Economy*, 113, 949–995.

Acemoglu, D. and A. Tahbaz-Salehi (2020): "Firms, Failures, and Fluctuations: The Macroeconomics of Supply Chain Disruptions," Tech. rep., National Bureau of Economic Research.

Amelkin, V. and R. Vohra (2019): "Strategic Formation and Reliability of Supply Chain Networks," *arXiv preprint arXiv:1909.08021*.

Antràs, P. (2005): "Incomplete Contracts and the Product Cycle," *American Economic Review*, 95, 1054–1073.

Atalay, E., A. Hortacsu, J. Roberts, and C. Syverson (2011): "Network Structure of Production," *Proceedings of the National Academy of Sciences*, 108, 5199–5202.

Baqaee, D. R. (2018): "Cascading Failures in Production Networks," *Econometrica*, 86, 1819–1838.

Baqaee, D. R. and E. Farhi (2019): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," *Econometrica*, 87, 1155–1203.

——— (2020): "Productivity and misallocation in general equilibrium," *Quarterly Journal of Economics*, 135, 105–163.

Barrot, J.-N. and J. Sauvagnat (2016): "Input Specificity and the Propagation of Idiosyncratic Shocks in Production Networks," *The Quarterly Journal of Economics*, 131, 1543–1592.

Bergstrom, T., L. Blume, and H. Varian (1986): "On the Private Provision of Public Goods," *Journal of Public Economics*, 29, 25–49.

[46]The unscaled reliability curve for complex production lies below the reliability curve of simple production. So complex production is more valuable but less reliable.

Bernstein, L. (1992): "Opting Out of the Legal System: Extralegal Contractual Relations in the Diamond Industry," *The Journal of Legal Studies*, 21, 115–157.

Bimpikis, K., O. Candogan, and S. Ehsani (2019a): "Supply Disruptions and Optimal Network Structures," *Management Science*, 65, 5504–5517.

Bimpikis, K., S. Ehsani, and R. Ilkiliç (2019b): "Cournot Competition in Networked Markets," *Management Science*, 65, 2467–2481.

Blanchard, O. and M. Kremer (1997): "Disorganization," *The Quarterly Journal of Economics*, 112, 1091–1126.

Blume, L., D. Easley, J. Kleinberg, R. Kleinberg, and É. Tardos (2011): "Network Formation in the Presence of Contagious Risk," in *Proceedings of the 12th ACM Conference on Electronic Commerce*, ACM, 1–10.

Boehm, C. E., A. Flaaen, and N. Pandalai-Nayar (2019): "Input Linkages and the Transmission of Shocks: Firm-Level Evidence from the 2011 Tōhoku Earthquake," *Review of Economics and Statistics*, 101, 60–75.

Boehm, J. and E. Oberfield (2020): "Misallocation in the Market for Inputs: Enforcement and the Organization of Production," *Quarterly Journal of Economics*, forthcoming.

Brummitt, C. D., K. Huremović, P. Pin, M. H. Bonds, and F. Vega-Redondo (2017): "Contagious Disruptions and Complexity Traps in Economic Development," *Nature Human Behaviour*, 1, 665.

Buldyrev, S. V., R. Parshani, G. Paul, H. E. Stanley, and S. Havlin (2010): "Catastrophic Cascade of Failures in Interdependent Networks," *Nature*, 464, 1025.

Cabrales, A., P. Gottardi, and F. Vega-Redondo (2017): "Risk Sharing and Contagion in Networks," *The Review of Financial Studies*, 30, 3086–3127.

Carvalho, V. M., M. Nirei, Y. U. Saito, and A. Tahbaz-Salehi (2020): "Supply Chain Disruptions: Evidence from the Great East Japan Earthquake," University of Cambridge.

Carvalho, V. M. and A. Tahbaz-Salehi (2019): "Production networks: A primer," *Annual Review of Economics*, 11, 635–663.

Carvalho, V. M. and N. Voigtländer (2014): "Input Diffusion and the Evolution of Production Networks," Tech. rep., National Bureau of Economic Research.

Chenery, H. B., S. Robinson, and M. Syrquin (1986): *Industrialization and Growth*, world Bank Washington.

Ciccone, A. (2002): "Input Chains and Industrialization," *The Review of Economic Studies*, 69, 565–587.

Craighead, C. W., J. Blackhurst, M. J. Rungtusanatham, and R. B. Handfield (2007): "The Severity of Supply Chain Disruptions: Design Characteristics and Mitigation Capabilities," *Decision Sciences*, 38, 131–156.

Dasaratha, K. (2021): "Innovation and Strategic Network Formation," *Review of Economic Studies*, forthcoming.

Datta, P. (2017): "Supply Network Resilience: a Systematic Literature Review and Future Research," *The International Journal of Logistics Management*.

Elliott, M. and B. Golub (2021): "Networks and economic fragility," .

Elliott, M., B. Golub, and M. O. Jackson (2014): "Financial Networks and Contagion," *American Economic Review*, 104, 3115–53.

Elliott, M., J. Hazell, and C.-P. Georg (2018): "Systemic Risk-Shifting in Financial Networks," *Available at SSRN 2658249*.

Erol, S. (2018): "Network Hazard and Bailouts," *Available at SSRN 3034406*.

Erol, S., F. Parise, and A. Teytelboym (2020): "Contagion in Graphons," *Available at SSRN*.

Erol, S. and R. Vohra (2018): "Network Formation and Systemic Risk," *Available at SSRN 2546310*.

Fafchamps, M. (2002): "Spontaneous Market Emergence," *Topics in Theoretical Economics*, 2.

Fafchamps, M. and B. Minten (1999): "Relationships and Traders in Madagascar," *The Journal of Development Studies*, 35, 1–35.

Gabaix, X. (2011): "The Granular Origins of Aggregate Fluctuations," *Econometrica*, 79, 733–772.

Gale, D. and S. Kariv (2009): "Trading in Networks: A Normal Form Game Experiment," *Americal Economic Journal: Microeconomics*, 1, 114–132.

Goyal, S. (2017): "Networks and markets," in *Advances in Economics and Econometrics: Volume 1: Eleventh World Congress*, Cambridge University Press, vol. 58, 215.

Graddy, K. (2006): "Markets: the Fulton Fish Market," *Journal of Economic Perspectives*, 20, 207–220.

Granovetter, M. (1973): "The Strength of Weak Ties," *American Journal of Sociology*, 78, 1360–1380.

——— (1985): "Economic Action and Social Structure: the Problem of Embeddedness," *American Journal of Sociology*, 91, 481–510.

Hall, R. E. (1979): "A Theory of the Natural Unemployment Rate and the Duration of Employment," *Journal of Monetary Economics*, 5, 153–169.

Hendricks, K. B. and V. R. Singhal (2003): "The Effect of Supply Chain Glitches on Shareholder Wealth," *Journal of operations Management*, 21, 501–522.

——— (2005a): "Association between Supply Chain Glitches and Operating Performance," *Management Science*, 51, 695–711.

——— (2005b): "An Empirical Analysis of the Effect of Supply Chain Disruptions on Long-Run Stock Price Performance and Equity Risk of the Firm," *Production and Operations Management*, 14, 35–52.

Jackson, M. O. (2008): *Social and Economic Networks*, Princeton University Press.

Jackson, M. O. and A. Pernoud (2019): "What Makes Financial Networks Special? Distorted Investment Incentives, Regulation, and Systemic Risk Measurement," *Distorted Investment Incentives, Regulation, and Systemic Risk Measurement (March 1, 2019).*

Jackson, M. O., T. Rodriguez-Barraquer, and X. Tan (2012): "Social Capital and Social Quilts: Network Patterns of Favor Exchange," *American Economic Review*, 102, 1857–97.

Jones, C. I. (2011): "Intermediate Goods and Weak Links in the Theory of Economic Development," *American Economic Journal: Macroeconomics*, 3, 1–28.

Jovanovic, B. (1987): "Micro Shocks and Aggregate Risk," *The Quarterly Journal of Economics*, 102, 395–409.

Kirman, A. P. and N. J. Vriend (2000): "Learning to be Loyal. A Study of the Marseille Fish Market," in *Interaction and Market structure*, Springer, 33–56.

König, M. D., A. Levchenko, T. Rogers, and F. Zilibotti (2019): "Aggregate Fluctuations in Adaptive Production Networks," Tech. rep., Mimeo.

Kranton, R. E. and D. F. Minehart (2001): "A theory of Buyer-Seller Networks," *American Economic Review*, 91, 485–508.

Kremer, M. (1993): "The O-ring Theory of Economic Development," *The Quarterly Journal of Economics*, 108, 551–575.

Leontief, W. W. (1936): "Quantitative Input and Output Relations in the Economic Systems of the United States," *The Review of Economic Statistics*, 105–125.

Levchenko, A. A. (2007): "Institutional Quality and International Trade," *The Review of Economic Studies*, 74, 791–819.

Levine, D. (2012): "Production Chains," *Review of Economic Dynamics*, 15, 271–282.

Liu, E. (2019): "Industrial Policies in Production Networks*," *The Quarterly Journal of Economics*, 134, 1883–1948.

McLain, S. (2021): "Auto Makers Retreat From 50 Years of 'Just in Time' Manufacturing," *Wall Street Journal*.

McLain, S., Matthews, C. M., and C. Paris (2021): "Everywhere You Look, the Global Supply Chain Is a Mess," *Wall Street Journal*.

Murdoch, J., T. Marsden, and J. Banks (2000): "Quality, Nature, and Embeddedness: Some Theoretical Considerations in the Context of the Food Sector," *Economic Geography*, 76, 107–125.

Murphy, K. M., A. Shleifer, and R. W. Vishny (1989): "Industrialization and the big push," *Journal of political economy*, 97, 1003–1026.

Nishiguchi, T. (1994): *Strategic Industrial Sourcing: The Japanese advantage*, Oxford University Press on Demand.

Norrman, A. and U. Jansson (2004): "Ericsson's Proactive Supply Chain Risk Management Approach After a Serious sub-Supplier Accident," *International Journal of Physical Distribution & Logistics Management*, 34, 434–456.

Nunn, N. (2007): "Relationship-Specificity, Incomplete Contracts, and the Pattern of Trade," *The Quarterly Journal of Economics*, 122, 569–600.

Oberfield, E. (2018): "A Theory of Input-Output Architecture," *Econometrica*, 86.

Okuno-Fujiwara, M. (1988): "Interdependence of industries, coordination failure and strategic promotion of an industry," *Journal of International Economics*, 25, 25–43.

Petersen, M. A. and R. G. Rajan (1994): "The Benefits of Lending Relationships: Evidence from Small Business Data," *The Journal of Finance*, 49, 3–37.

——— (1995): "The Effect of Credit Market Competition on Lending Relationships," *The Quarterly Journal of Economics*, 110, 407–443.

Rodrik, D. (1996): "Coordination failures and government policy: A model with applications to East Asia and Eastern Europe," *Journal of International Economics*, 40, 1–22.

Rosenstein-Rodan, P. N. (1943): "Problems of industrialisation of eastern and south-eastern Europe," *The economic journal*, 53, 202–211.

Rostek, M. and M. Weretka (2015): "Dynamic Thin Markets," *The Review of Financial Studies*, 28, 2946–2992.

Sadler, E. (2020): "Diffusion Games," *American Economic Review*, 110, 225–270.

Scheinkman, J. A. and M. Woodford (1994): "Self-Organized Criticality and Economic Fluctuations," *The American Economic Review Paper and Proceedings*, 84, 417–421.

Slutsken, H. (2018): "Four million parts, 30 countries: How an Airbus A380 comes together," *CNN*.

Snyder, L. V., Z. Atan, P. Peng, Y. Rong, A. J. Schmitt, and B. Sinsoysal (2016): "OR/MS Models for Supply Chain Disruptions: A Review," *IIE Transactions*, 48, 89–109.

Supply Chain Quarterly (2018): "Supply Chain Disruptions Hit Record High," *Supply Chain Quarterly*, `https://www.supplychainquarterly.com/news/20181210-supply-chain-disruptions-hit-record-high/`.

Talamàs, E. and R. Vohra (2020): "Go Big or Go Home: A Free and Perfectly Safe but Only Partially Effective Vaccine Can Make Everyone Worse Off," *Games and Economic Behavior*, 122, 277–289.

Tang, L., K. Jing, J. He, and H. E. Stanley (2016): "Complex Interdependent Supply Chain Networks: Cascading Failure and Robustness," *Physica A: Statistical Mechanics and its Applications*, 443, 58–69.

The Economist (2006): "When the Chain Breaks," .

Tintelnot, F., A. K. Kikkawa, M. Mogstad, and E. Dhyne (2018): "Trade and Domestic Production Networks," Tech. rep., National Bureau of Economic Research.

Uzzi, B. (1997): "Social Structure and Competition in Interfirm Networks: The Paradox of Embeddedness," *Administrative Science Quarterly*, 35–67.

Yang, Q., C. Scoglio, and D. Gruenbacher (2019): "Discovery of a Phase Transition Behavior for Supply Chains against Disruptive Events," *arXiv preprint arXiv:1908.02616*.

Appendix A. Formal construction of the supply network

We now formally construct the random supply network which was introduced in Sections 2.1 and 2.2; this construction provides the foundation for calculating the set of functional firms, etc.

Endow each $\mathcal{V}_i$ with the Borel σ-algebra and a scaling of Lebesge measure, denoted λ, and let $\mathcal{V}$ be the disjoint union of these spaces, with total measure 1. Fix positive integers m and n, as well as a distribution μ over the nonnegative integers. A (symmetric) *potential supply network with parameters m, n, μ* is a random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ satisfying the following properties.

- Its nodes are the set $\mathcal{V}$.
- Edges are ordered pairs (v, v') where $v' = (j, f')$ for $j \in I(i)$—the meaning is that v can potentially source from v'. We depict such an edge as an arrow from v to v'.
- The measure of nodes v with $d(v) = d'$ is $\mu(d')$.
- Consider any $v \in \mathcal{V}_i$ with $d(v) > 0$.
  - For each $j \in I(i)$ there are n edges (v, v') to n distinct varieties $v' \in \mathcal{V}_j$. For any variety, define its neighborhood $N_v = \{v' : (v, v') \in \mathcal{E}\}$.
  - The elements of $N_v \cap \mathcal{V}_j$ are independently drawn from an atomless distribution over $\mathcal{V}_j$ conditioned on $d(v') = d(v) - 1$.
  - For any countable set of varieties $\widehat{\mathcal{V}}$, its neighborhoods $(N_v)_{v \in \widehat{\mathcal{V}}}$ are independent.

Now fix *relationship strengths* $(x_v)_{v \in \mathcal{V}}$.[47] Define $\mathcal{G}'$ to be a random subgraph of $\mathcal{G}$ in which each edge from a node v of positive depth is kept independently, with probability x_v. More formally, define for every edge vv' a random variable $O_{vv'} \in \{0, 1\}$ (whether the edge is operational) such that

- $\mathbf{P}[O_{vv'} = 1 \mid \mathcal{G}] = x_v$ for every $vv' \in \mathcal{G}$ and,
- for any countable subset E of edges in $\mathcal{G}$, the random variables $(O_e)_{e \in E}$ are independent conditional on $\mathcal{G}$.

The out-neighborhood of the depth-0 varieties in $\mathcal{G}'$ is $\bigcup_{j \in I(i)} \mathcal{V}_j$, since these can source from anyone.

A subset $\widehat{\mathcal{V}} \subseteq \mathcal{V}$ is defined to be *consistent* if, for each $v \in \widehat{\mathcal{V}}$ the following holds: for each product j that $v = (i, f)$ requires as inputs ($\forall\, j \in I(i)$), there is an operational edge $(v, v') \in \mathcal{G}'$) with $v' \in \widehat{\mathcal{V}}$. There may be many consistent sets, but by Tarki's theorem, there will be a maximal one, $\mathcal{V}'$, which is a superset of any other consistent set. For any given variety, this can be found by the simple iterative procedure in Section 2.3.

Since any countable set of edges is independent, we can make computations about the relevant marginal probabilities in our model (e.g., reliability of any variety) as we would if there only one tree.

Appendix B. Proofs of main results

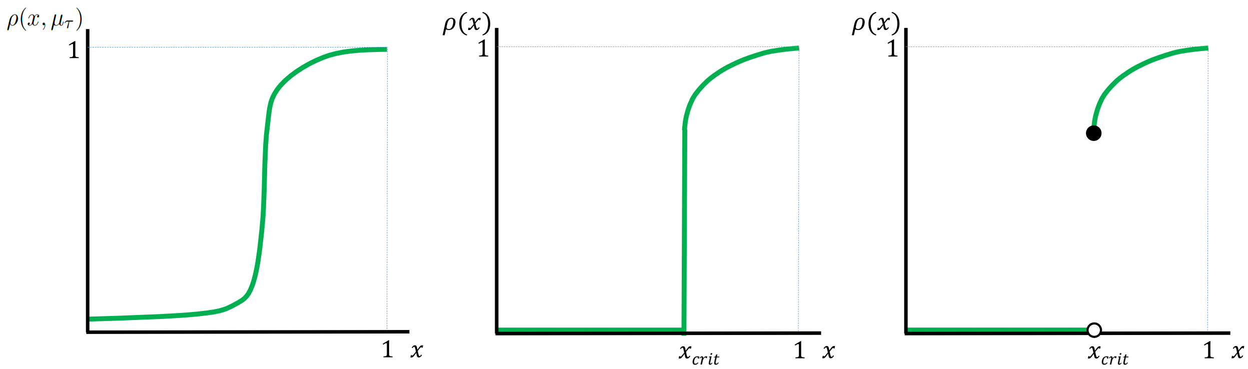
B.1. **The reliability curve: Basic precipice results.** The first building block of our analysis is characterizing the shape of the reliability curve $x \mapsto \rho(x, \mu_\tau)$ when τ is large (so that depths are large). In Figure 10, we reproduce Figure 3 (adding one additional panel), which will help us describe what we are doing. First, we define a correspondence that will be central to our analysis. It is a limit of the functions $x \mapsto \rho(x, \mu_\tau)$, in the sense that their graphs converge to the graph of the correspondence $\rho(x)$ as $\tau \to \infty$ (see panels (A) and (B)). More formally:

Definition 5. Let $\rho : [0, 1] \rightrightarrows [0, 1]$ be a correspondence such that[48]

(i) for any $r \in \rho(x)$, there is a sequence $\{x_\tau\}_{\tau=1}^{\infty} \to x$ such that $\lim_{\tau \to \infty} \rho(x_\tau, \mu_\tau) = r$;

[47] Here $v \mapsto x_v$ is a given measurable function.

[48] Equivalently, we can define ρ by saying that it is a correspondence such that the graphs of the functions $\rho(\cdot, \mu_\tau) : [0, 1] \to [0, 1]$ converge to the graph of ρ (in the Hausdorff set-distance metric).



(A) (B) (C)

FIGURE 10. Panel (A) shows how reliability varies with the investment level x when supply chains have a finite depth. Panel (B) shows the limit correspondence of the relationships between reliability r and the investment level x as the depth gets large. Panel (C) shows how reliability varies with the investment level x for infinite depth.

(ii) for any sequence $\{x_\tau\}_{\tau=1}^{\infty} \to x$ we have $\lim_{\tau\to\infty} \rho(x_\tau, \mu_\tau) \in \rho(x)$.

We will show that this limit correspondence ρ exists and is uniquely defined, and that it has the shape sketched in panel (B): zero until some point, with a vertical rise up to a certain critical level of reliability, followed by a concave ascent to reliability 1 at $x = 1$. With these results in hand, this subsection will culminate in the proofs of Proposition 1 and Proposition 2.

Our main result on the reliability correspondence is:

Proposition 6 (Shape of the limit reliability correspondence ρ)**.** Let the complexity of production be $m \geq 2$ and the number of potential suppliers for each input be $n \geq 2$. Then there is a unique ρ satisfying Definition 5, and it has the following properties. There exists an $x_{\text{crit}} \in (0, 1)$ such that

(i) $\rho(x)$ is single-valued for all $x \neq x_{\text{crit}}$;
(ii) $\rho(x) = 0$ for all $x < x_{\text{crit}}$;
(iii) there is a value $0 < \overline{r}_{\text{crit}} < 1$ such that $\rho(x_{\text{crit}}) = [0, \overline{r}_{\text{crit}}]$;
(iv) $\rho(x)$ is strictly increasing in x for all $x > x_{\text{crit}}$;
(v) $\lim_{x \downarrow x_{\text{crit}}} \rho'(x) = \infty$.

Sections B.1.1 and B.1.2 are devoted to proving this result. A closely related (discontinuous) function, sketched in panel (c), will play a role in this analysis, as we describe next. Once this is done, we will deduce Propositions 1 and 2.

B.1.1. *The reliability function with infinite depth: A tool to study ρ.* In order to prove Proposition 6 it is helpful to define and characterize some properties of a certain function $\widehat{\rho} : [0, 1] \to [0, 1]$, sketched in Figure 10(C). It will turn out that for all $x \neq x_{\text{crit}}$, $\rho(x) = \widehat{\rho}(x)$. We define $\widehat{\rho}(x)$ as the largest r solving the following equation

$$\widehat{r} = (1 - (1 - x\widehat{r})^n)^m. \tag{6}$$

(For intuition, note that this is equation (2), which relates reliability at depths d and $d-1$, but here with the same reliability r on both sides. This is an intuitive condition "at depth infinity" as reliability is not decreased by adding one layer of depth.)

Lemma 2, which is proved in Section SA1.1 of the Supplementary Appendix, identifies several key properties of $\widehat{\rho}(x)$.

Lemma 2. Suppose the complexity of the supply network is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \tag{7}$$

Then there are values $x_{\text{crit}}, \overline{r}_{\text{crit}} \in (0, 1]$ such that:

(i) $\widehat{\rho}(x) = 0$ for all $x < x_{\text{crit}}$;
(ii) $\widehat{\rho}$ has a (unique) point of discontinuity at x_{crit};
(iii) $\widehat{\rho}$ is strictly increasing for $x \geq x_{\text{crit}}$;
(iv) the inverse of $\widehat{\rho}$ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[\overline{r}_{\text{crit}}, 1]$, where $\overline{r}_{\text{crit}} = \widehat{\rho}(x_{\text{crit}})$;
(v) χ is positive and quasiconvex on the domain $(0, 1]$;
(vi) $\chi'(\overline{r}_{\text{crit}}) = 0$.

The proof of Lemma 2 is in Section SA1.1 of the Supplementary Appendix. While the manipulations to establish these properties are a bit involved, they amount to studying the function $\widehat{\rho}$ and the pseudo-inverse of it that we have defined, χ, using calculus. Figure 11(a) depicts χ as a function of r.[49]

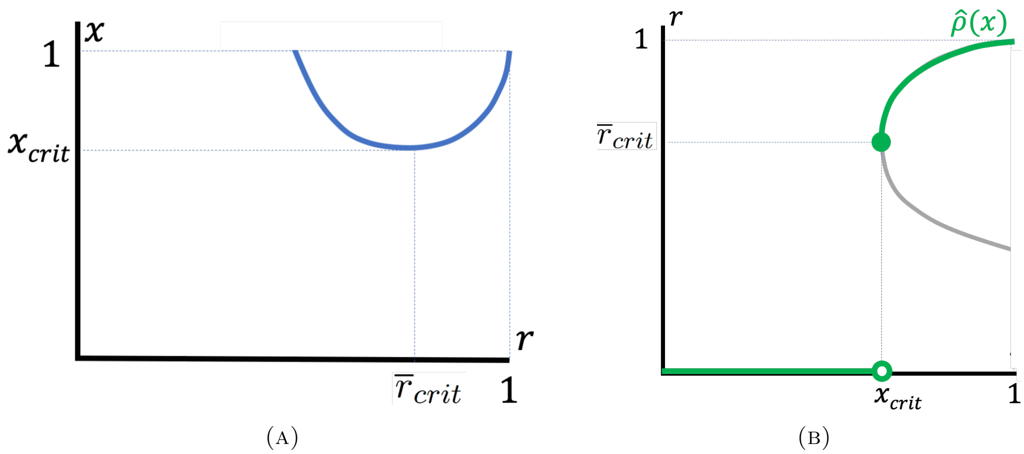


FIGURE 11. Panel (A) plots the function χ as r varies, and then in Panel (B) we show how switching the axes and taking the largest r value on the graph (corresponding to the largest solution of equation (6)) generates $\widehat{\rho}(x)$.

Recall the functions $\widetilde{\rho}(x, d)$: consider a depth-d tree where each firm in each tier requires m kinds of inputs and has n potential suppliers of each input. We denote by $\widetilde{\rho}(x, d)$ the probability of successful production at the most-downstream node of a depth-d tree with these properties. This is defined as

$$\widetilde{\rho}(x, d) = (1 - (1 - x\widetilde{\rho}(x, d-1))^n)^m$$

with $\widetilde{\rho}(x, 0) = 1$, since the most-upstream tier nodes obtain their inputs without the possibility of disruption.

It will be useful throughout that $\widetilde{\rho}(x, d)$ and its derivative converge uniformly to the function $\widehat{\rho}(x)$ everywhere except near x_{crit}. Indeed, this will direclty imply that reliability curve and the marginal returns to investment in our model also converge (at all points but x_{crit}) to their corresponding values in the "infinite-depth" model that defines $\widehat{\rho}$. The following lemma formalizes these statements.

Lemma 3.

(i) For all $d \geq 1$, the function $x \mapsto \widetilde{\rho}(x, d)$ defined for $x \in (0, 1)$ is strictly increasing and infinitely differentiable.
(ii) On any compact set excluding x_{crit}, the sequence $(\widetilde{\rho}(x, d))_{d=1}^{\infty}$ converges uniformly to $\widehat{\rho}(x)$.
(iii) On any compact set excluding x_{crit}, the sequence $(\rho(x, \mu_\tau))_{\tau=1}^{\infty}$ converges uniformly to $\widehat{\rho}(x)$.

Proof. The sequence $(\widetilde{\rho}(\cdot, d))_{d=1}^{\infty}$ is a monotone sequence of increasing, infinitely differentiable functions,[50] converging pointwise to $\widehat{\rho}$. We know that $\widehat{\rho}$ is continuous on any compact set excluding x_{crit}. Therefore, by Dini's theorem, the functions $\widetilde{\rho}(\cdot, d)$ converge uniformly to $\widehat{\rho}$.

[49]Since by definition the *largest* r satisfying (6) is the one that determines $\widehat{\rho}(x)$, it follows that the increasing part of the function, where $r \in [\overline{r}_{\text{crit}}, 1]$ is the part relevant for determining equilibrium reliability—see Figure 11(b), where the light gray branch is not part of $\widehat{\rho}$.

[50]From eq. (2), it is clear that increasing d decreases $\widetilde{\rho}(x, d)$, while increasing x increases it. Differentiability is straightforward from the iterative definition.

Note the functions $\rho(\cdot, \mu_\tau)$ are strictly increasing and infinitely differentiable, since they are the averages of such functions. Moreover, they clearly converge pointwise to $\widehat{\rho}(x)$: since all the $\widehat{\rho}(\cdot, d)$ are uniformly bounded, the vanishing probability mass on low-d realizations makes only a negligible contribution to the value of $\rho(x, \mu_\tau)$ for large τ. From this the conclusions of Lemma 3 apply equally well to the sequence of functions $(\rho(\cdot, \mu_\tau))_{\tau=1}^{\infty}$. □

B.1.2. *Proof of Proposition 6.* We will now establish a close relationship between $\widehat{\rho}$ and our correspondence of interest, ρ, which will allow us to use Lemmas 2 and 3 to prove Proposition 6.

Define $\rho(x) = \{\widehat{\rho}(x)\}$ for $x \neq x_{\text{crit}}$. By Lemma 3(iii) $\rho(x, \mu_\tau) \to \widehat{\rho}(x)$ as $\tau \to \infty$ for any $x \neq x_{\text{crit}}$. Thus Part (ii) of Definition 5 is satisfied. Since $\rho(x)$ is single-valued for $x \neq x_{\text{crit}}$, Part (i) of Definition 5 also holds. Lemma 2 then implies points (i), (ii), (iv) and (v) of Proposition 6.

Next, define $\rho(x_{\text{crit}}) = [0, \overline{r}_{\text{crit}}]$. We will show that the remaining conditions of Definition 5 (those pertaining to x_{crit}) also hold. First, we will check Part (i) of Definition 5 for $r \in \rho(x_{\text{crit}})$. For any $r \in [0, \overline{r}_{\text{crit}}]$, we can construct a sequence $\{x_\tau\}_{\tau=1}^{\infty} \to x_{\text{crit}}$ such that $\lim_{\tau\to\infty} \rho(x_\tau, \mu_\tau) = r$. To see this, simply note that for any $r \in (0, \overline{r}_{\text{crit}}]$ and any τ, there is an x_τ such that $\rho(x_\tau, \mu_\tau) = r$ since $\rho(x, \mu_\tau)$ is a continuous and increasing function of x whose image is $[0, 1]$. Moreover, since for any $x < x_{\text{crit}}$, $\lim_{\tau\to\infty} \rho(x, \mu_\tau) = 0$ and for any $x > x_{\text{crit}}$ there exists $\epsilon > 0$ such that $\lim_{\tau\to\infty} \rho(x, \mu_\tau) = \overline{r}_{\text{crit}} + \epsilon$, it follows that these x_τ must indeed converge to x_{crit}. In the case of $r = 0$, there exists a sequence $\{x_\tau\}_{\tau=1}^{\infty} \to x_{\text{crit}}$ such that $\rho(x_\tau, \mu_\tau) \downarrow 0$ since for any $x < x_{\text{crit}}$, $\rho(x, \mu_\tau) \downarrow 0$.

Finally, we show that every sequence $\{x_\tau\}_{\tau=1}^{\infty} \to x_{\text{crit}}$ satisfies $\lim_{\tau\to\infty} \rho(x_\tau, \mu_\tau) \in [0, \overline{r}_{\text{crit}}]$ (Part (ii) of Definition 5). This amounts to showing that the limit is at most r_{crit}. Suppose otherwise, that such a limit is $r' > r_{\text{crit}}$. Let x' be such that $\widehat{\rho}(x') = r'$, which exists by Lemma 2. Note for all $x_\tau < x'$, we have $\widehat{\rho}(x_\tau) < r' - \epsilon$ for some positive ϵ, and so $\rho(x_\tau, \mu_\tau) < r'$ for sufficiently large τ; this is a contradiction to the hypothesis about the limit being r'.

These claims together establish the remaining content of the claim that the ρ we have defined is the limit satisfying Definition 5. That implies part (iii) of Proposition 6. □

B.1.3. *Proof of Proposition 1 (Discontinuity in reliability) using the limit reliability correspondence.* We can now use the results established to prove the proposition about the precipice.

Part (i): The fact that $\rho(x, \mu_\tau) \to 0$ for $x < x_{\text{crit}}$ follows from Definition 5 and Proposition 6(ii).

Part (ii): From Proposition 6(iii) and (iv), it follows that $\rho(x) > \overline{r}_{\text{crit}}$ for any $x > x_{\text{crit}}$ and thus, for τ large enough, $\rho(x, \mu_\tau) > \overline{r}_{\text{crit}} > 0$. □

B.2. **Proof of Proposition 2 (Social planner's solution).** Now we can use the properties of the reliability curve to prove our result on the social planner's solution. Recall that $x^{SP}(\kappa, \mu_\tau)$ is the set of all values of x maximizing the planner's objective.

First consider the following "limit" planner's problem, defined for $x \in [0, 1]$:

$$\max \left[h(r) - \frac{1}{\kappa} c_P(x) \right] \text{ subject to } r \in \rho(x). \tag{8}$$

We claim that, if $\delta > 0$ is chosen small enough, then any solution has $x \notin [x_{\text{crit}} - \delta, x_{\text{crit}} + \delta]$. To rule out solutions with $x \in [x_{\text{crit}} - \delta, x_{\text{crit}})$, note that by Property B and continuity of c_P that the cost $c_P(x)$ is positive in that interval (if δ is fixed at a small enough value) while $\rho(x)$ is zero by Proposition 1, so that $x = r = 0$ does better. To rule out solutions with $x \in [x_{\text{crit}}, x_{\text{crit}} + \delta]$, first observe that for $x = x_{\text{crit}}$, the only r that can be a solution is $r = r_{\text{crit}}$, so we may assume the solution lies on the graph of $\widehat{\rho}$. Next, note that if δ is chosen small enough, then for any κ, we have $\widehat{\rho}'(x) > \frac{1}{\kappa} c_P(x)$ in the interval $[x_{\text{crit}}, x_{\text{crit}} + \delta]$ since $\widehat{\rho}'(x)$ tends to $+\infty$ as $x \downarrow x_{\text{crit}}$ by Lemma 3.

Let $\kappa_{\text{crit}} > 0$ be the minimum value of κ such that that (8) has a solution with $r > 0$. This κ_{crit} exists because for large enough κ we have that the maximand is positive at $x = x_{\text{crit}}$, and so there must be a solution with positive r; similarly, κ_{crit} is positive since for small κ, the maximand is negative for all $x \geq x_{\text{crit}}$.[51] By the previous paragraph, any solution at $\kappa = \kappa_{\text{crit}}$ satisfies $x > x_{\text{crit}}$. Since in the domain $x > x_{\text{crit}}$, we have that $\rho(x)$ is concave while c_P is convex, for $\kappa \geq \kappa_{\text{crit}}$ there is a unique, strictly positive solution $(x(\kappa), r(\kappa))$

[51]Since $c_P(x_{\text{crit}}) > 0$ and c_P is convex, $c_P(x) > 0$ for all $x \geq x_{\text{crit}}$. Thus by making κ small enough, the costs exceed the bounded benefits. We can say "minimum" rather than "infimum" in the definition of κ_{crit} because the correspondence ρ is continuous, as is c_P.

of (8) and both x and r are increasing in κ. For $\kappa < \kappa_{\text{crit}}$, we have that $\rho(x) = 0$ at the optimum and hence cost is zero (otherwise $r = x = 0$ would do better). Thus, by what we have said above,

(i) for all $\kappa < \kappa_{\text{crit}}$, all solutions of (8) have $x \le x_{\text{crit}} - \delta$, have cost equal to 0, and yield reliability 0;
(ii) for all $\kappa > \kappa_{\text{crit}}$, all solutions of (8) have $x \ge x_{\text{crit}} + \delta$ and yield reliability $r \ge r_{\text{crit}}$;
(iii) for $\kappa = \kappa_{\text{crit}}$, all solutions of (8) are outside the interval $[x_{\text{crit}} - \delta, x_{\text{crit}} + \delta]$ and reliability is either 0 or strictly above r_{crit}.

Returning to the main model rather than the limit case above: for any κ and τ, consider the following rewriting of the planner's problem:

$$\max \left[h(r) - \frac{1}{\kappa} c_P(x) \right] \text{ subject to } r = \rho(x; \mu_\tau). \tag{9}$$

For any κ, it is straightforward to deduce from the definition of the correspondence ρ that a sequence of solutions $(x(\tau), \rho(x; \mu_\tau))$ of (9) must converge to a solution of (8) as $\tau \to \infty$.[52] Thus, choosing ϵ small enough and using the convergence of $\rho(x; \mu_\tau)$ to $\rho(x)$ per Definition 5 (along with the properties of ρ established in Proposition 1), the conclusions of the present proposition follow. □

B.3. **Best responses and equilibrium behavior: Oveview and preliminaries.** Our goal now is to work toward Theorem 1 on the ordered regimes and the properties of equilibrium.

For notational convenience, when analyzing equilibria, we will multiply through all firms' profit functions by κ, so that a firm's cost is $c(x_{if} - \underline{x})$ and its gross profit is $\kappa g(r) P(x_{if}; r)$. Note this does not change any best responses. We maintain this convention throughout the remainder of the proofs, and in the Supplementary Appendix.

In Figure 12, we reproduce part of Figure 5, which will guide our analysis. Before beginning, we outline the plan. At a high level, it amounts to bringing the best-response curve into the picture along with the precipice graph, and ultimately formalizing the graphical intuitions in Figure 12 concerning what happens to the best-response curve as κ moves around. Throughout, a recurring strategy is to prove a result for the game "at the $\tau = \infty$" limit (in a sense we make precise), and then extend the conclusion to large values of τ, where most of the mass is on deep supply trees.

To study this $\tau = \infty$ limit, we establish some notation. First, define

$$\widehat{P}(x_{if}; r) = (1 - (1 - x_{if} r)^n)^m. \tag{10}$$

This is the probability with which if is able to produce, as a function of its investment choice x_{if} given that all its suppliers have reliability r. This is simpler than our finite-τ problem because all firms are exactly symmetric (which is only approximately true when τ is large but finite).

Define

$$MB(x_{if}; r, \kappa) = \kappa g(r) \frac{\partial \widehat{P}(x_{if}; r)}{\partial x_{if}} \tag{11}$$

$$MC(x_{if}) = c'(x_{if} - \underline{x}). \tag{12}$$

These are the marginal benefit and marginal cost, respectively, to a firm of investing in relationship strength for the supply network "at the limit."

A first step in the analysis is making best responses at $\tau = \infty$ tractable by expressing them as the solution to a first-order condition equating marginal benefits and marginal costs, which is unique under suitable assumptions. This is done in Section B.3.1. Second, we show in Section B.3.2 that if we are interested in equilibria that are undominated in terms of social surplus, we may focus on the higher points of intersection in Figure 12. Third, we show in Section B.3.3 that in the limit model, there is a unique intersection between the two curves above reliability $\overline{r}_{\text{crit}}$, as in Figure 12(d), with the best-response curve sloping down in r, which makes the comparative statics work as in the sketches.

Together, these ingredients set up the proof of Theorem 1 in Section B.4, allowing us to formalize the graphical intuition. The remaining subsections carry out this plan.

B.3.1. *Proof of Lemma 1 (Sufficient condition for unique interior local maximum of the firm's objective).* We first study the firm's problem "at $\tau = \infty$." Afterward, we establish that the same uniqueness property holds for large, finite τ.

[52] This is a version of the Theorem of the Maximum on upper hemicontinuity of optimization.

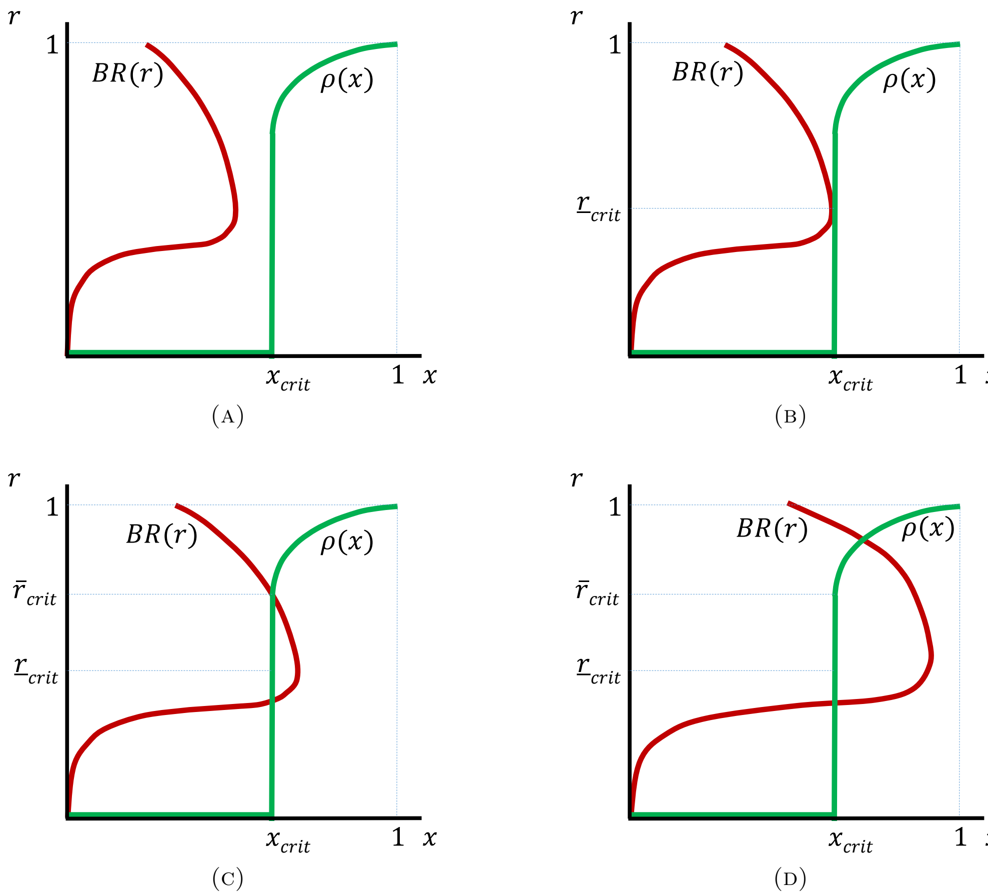


FIGURE 12. Panel (A) shows an equilibrium for $\kappa < \underline{\kappa}$. Panel (B) shows an equilibrium with $\kappa = \underline{\kappa}$. Panel (C) shows an equilibrium with $\kappa = \overline{\kappa}$. Panel (D) shows an equilibrium with $\kappa > \overline{\kappa}$. Panel (E) plots how equilibrium reliability varies with κ. Panel (F) shows reliability following an arbitrarily small negative shock to institutional quality $\underline{x}$ as κ varies.

For the extended domain $x_{if} \in [0, 1/r]$, we define

$$Q(x_{ik}; r) := \frac{\partial}{\partial x_{if}} \widehat{P}(x_{if}; r), \tag{13}$$

which can be calculated to be $Q(x_{ik}; r) = mn(1 - (1 - x_{if}r)^n)^{m-1}(1 - x_{if}r)^{n-1}r$.

We will need two steps to prove Lemma 1. The first step consists of establishing Lemma 4 on the basic shape of $Q(x_{if}; r)$. Figure 13 illustrates the shape of $Q(x_{if}; r)$ implied by Lemma 4.

Lemma 4. Fix any $m \geq 2$, $n \geq 2$, and $r \geq \underline{r}_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n, and x) such $0 \leq x_1 < x_2 < 1/r$ and so that:

0. $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$;
1. $Q(x_{if}; r)$ is increasing and convex in x_{if} on the interval $[0, x_1]$;
2. $Q(x_{if}; r)$ is increasing and concave in x_{if} on the interval $(x_1, x_2]$;
3. $Q(x_{if}; r)$ is decreasing in x_{if} on the interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

The proof of Lemma 4 is in Section SA1.2 of the Supplementary Appendix.

We now complete the proof of Lemma 1 by setting $\widehat{x} = x_1$. By Lemma 4(4), the interval $(\widehat{x}, x_{\text{crit}})$ is non-empty. Thus we just need to show that Assumption 1 (the sufficiency of the first-order condition for interior optima) is satisfied when $\underline{x} \in (\widehat{x}, x_{\text{crit}})$. Note that by Property B′, we have that $c'(0) = 0$ and c' is weakly increasing and weakly convex otherwise. Since, by Lemma 4, $P'(x_{if}; x)$ is first concave and increasing (possibly for the empty interval) and then decreasing (possibly for the empty interval) over the

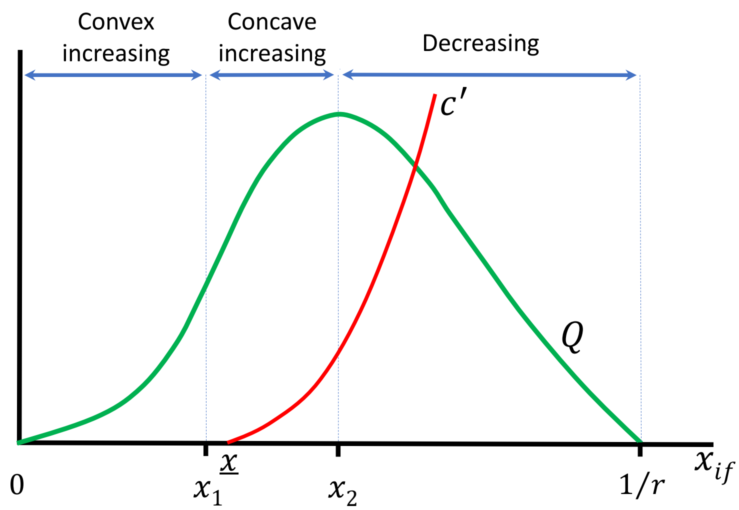


FIGURE 13. The shape of $Q(x_{if}, x)$ in green and $c'(x_{if} - \underline{x})$ in red. There can be only one intersection between the two curves, which corresponds to the maximizer of $\Pi(x_{if}; r)$. (Here we normalized $\kappa g(r) = 1$ for simplicity, but the illustration remains valid if the green cruve is scaled.)

range $x_{if} \in [\underline{x}, 1]$, it follows that there is at most a single intersection between the curves $P'(x_{if}; x)$ and $c'(x_{if} - \underline{x})$. This intersection corresponds to the first-order condition $Q(x_{if}; r) = c'(x_{if} - \underline{x})$, yielding the unique maximizer of $\Pi(x_{if}; x)$, as illustrated in Figure 13. If such an intersection not exist, $y_{if} = 0$ is a local and global maximizer of the profit function.

Now we study the quantities introduced above, but for finite a τ and show that when τ is large enough, we still obtain a unique solution to the firm's problem.

Recall that

$$P(x_{if}; x, \mu_\tau) = \mu_\tau(0) + \sum_d \mu_\tau(d)(1 - (1 - x_{if}\widetilde{\rho}(x, d-1))^n)^m.$$

Define

$$Q(x_{if}; x, \mu_\tau) = \frac{\partial}{\partial x_{if}} P'(x_{if}; x, \mu_\tau) = \sum_d \mu_\tau(d) mn(1-(1-x_{if}\widetilde{\rho}(x, d-1))^n)^{m-1}(1-x_{if}\widetilde{\rho}(x, d-1))^{n-1}\widetilde{\rho}(x, d-1).$$

Recall that for all $x \neq x_{\text{crit}}$ we have

$$\widetilde{\rho}(x, d) \to_d \widehat{\rho}(x) \text{ and } \rho(x, \mu_\tau) \to_\tau \widehat{\rho}(x).$$

It follows from this and the expression for $Q(x_{if}; x, \mu_\tau)$ above that

$$P(x_{if}; x, \mu_\tau) \to_\tau \widehat{P}(x_{if}; \widehat{\rho}(x)) \text{ and } Q(x_{if}; x, \mu_\tau) \to_\tau Q(x_{if}; \widehat{\rho}(x)).$$

It follows that marginal costs and marginal benefits in the $\tau \to \infty$ limit are arbitrarily close to the functions studied above. We then conclude that for τ large enough, there will also be at most a single intersection between the curves $Q(x_{if}; x, \mu_\tau)$ and $c'(x_{if} - \underline{x})$ and thus the results stated in the limit also hold for τ large enough. □

B.3.2. *A lemma on symmetric undominated equilibria.* This subsection establishes that the symmetric equilibrium maximizing social surplus is the one with greatest relationship strengths x and highest reliability. In other words, it is the higher intersection in Figure 12 when there are several. Normalizing $\kappa = 1$ without loss of generality, recall our notation that at a symmetric equilibrium with reliability r, gross output is $h(r)$ and gross profits are $g(r)$.

Lemma 5. Consider two symmetric equilibria with relationship strengths $x_1 < x_2$ and reliabilities $r_1 < r_2$. Then $h(r_2) - c_P(x_2) > h(r_1) - c_P(x_1)$. That is, the higher-investment equilibrium has the greater net social surplus.

Proof. Let $V(x) = h(\rho(x, \mu)) - c_P(x)$. We will first show that V is increasing at $x = x_2$, and then use this to deduce the conclusion.

Consider an outcome with firm investments x_V and a reliability of r. Note that the gross profit per functional firm is $g(r)$ so that the gross profit integrated across all firms (functional or nonfunctional) is

$$\mathrm{GP} = rg(r).$$

We will write net social surplus as follows, and then split it up in to consumer and producer surplus:

$$\begin{aligned} V &= h(r) - \int_{\mathcal{V}} c(x_v)dv \\ &= [h(r) - \mathrm{GP}] + \left[\mathrm{GP} - \int_{\mathcal{V}} c(x_v)dv\right] \\ &= \underbrace{[h(r) - rg(r)]}_{\mathrm{CS}} + \mathrm{PrS}\,. \end{aligned}$$

Here we define total producer surplus, PrS as gross profits net of investment costs.

Now, it will be useful to have two different expressions for producer surplus, which are identically equal:

$$\mathrm{PrS}_1 = rg(r) - \int_{\mathcal{V}} c(x_v)dv \tag{14}$$

$$\mathrm{PrS}_2 = \int_{\mathcal{V}} [\Pi_v(x_v; r) - c(x_v)]dv. \tag{15}$$

Now make all the variables—x_v's and r's—functions of a parameter t, according to $x_v = x + t$ for small t, and r is a function of the x_v (and thus of t. We want to totally differentiate V in t. We claim the derivative is positive. First consider

$$\frac{d}{dt}\,\mathrm{CS} = \frac{d}{dt}\left[h(r) - rg(r)\right].$$

Now we turn to producer surplus. Here, we need to totally differentiate in t. The endogenous variables are the continuum of x_v's and r:

$$\frac{d}{dt}\,\mathrm{PrS} = \frac{dr}{dt}\cdot\frac{\partial}{\partial r}\,\mathrm{PrS} + \frac{dx_v}{dt}\int_{\mathcal{V}}\frac{\partial}{\partial x_v}\,\mathrm{PrS}\,.$$

Now, the key idea of the argument is that we use the PrS_1 expression for PrS in the first summand, and the PrS_2 expression for PrS in the second summand, which is legimiate since the expressions are identically equal. The second expression evaluates to 0 because for a fixed v, we have

$$\frac{\partial}{\partial x_v}\,\mathrm{PrS}_2 = \frac{\partial}{\partial x_v}[\Pi_v(x_v; r) - c(x_v)],$$

which is 0 by the firm's first-order condition.

Using $V = \mathrm{CS} + \mathrm{PrS}$ and our calculations above,

$$\frac{dV}{dt} = \frac{dr}{dt}h'(r) = \frac{d}{dx}\rho(x,\mu)h'(\rho(x,\mu)).$$

Both factors are clearly positive, so V is increasing in t.

Note that for $x \geq x_{\mathrm{crit}}$, we have that h is concave by Property A and increasing in x while $c_P(x)$ is convex, so

$$h(\rho(x,\mu)) - c_P(x)$$

is single-peaked. The above argument implies both equilibria are socially inefficient, and thus social surplus is increasing in x at both of them. This combined with single-peakedness shows that the one with higher relationship strength is more efficient. □

B.3.3. *A limit uniqueness result for equilibria above $\overline{r}_{crit}$.* In this section we prove an important lemma that will be used to ensure that the relationship strengths and reliability achieved at symmetric undominated equilibria are uniquely determined. As before, we study the "$\tau = \infty$" model as a stepping stone to our large-τ results. Visually, the lemma establishes that there is at most one intersection in Figure 12 that lies above $\overline{r}_{\mathrm{crit}}$.

To motivate the statement, consider the question: when can there be an equilibrium outcome with $r \geq \overline{r}_{\mathrm{crit}}$ for the limit payoffs at $\tau = \infty$? Recall the definitions of marginal benefits and marginal costs from (11) and (12) above. Under Assumption 1, an interior firm optimum is characterized by the first order condition (recall equations (11) and (12) for the definitions)

$$MB(x; r, \kappa) = MC(x), \tag{OI}$$

which we have labeled OI, for optimal investment. In addition, an outcome of the $\tau = \infty$ model also satisfies $r = \widehat{\rho}(x)$ for $r \geq \overline{r}_{\text{crit}}$: the reliability is the one induced by firms' choices of relationship strengths. Recalling the definition of χ from Lemma 2, this entails $x = \chi(r)$. Then we have the equation

$$MB(\chi(r); r, \kappa) = MC(\chi(r)).$$

We will show that (for some $\epsilon > 0$) there is at most one solution (x, r) satisfying $r \in [\overline{r}_{\text{crit}} - \epsilon, 1]$ and $x = \chi(r)$ simultaneously. This implies that there is only one solution to the first-order conditions above $\overline{r}_{\text{crit}}$, and the fact that the necessary condition cannot hold even for slightly lower values of $\overline{r}_{\text{crit}}$ will be technically useful later.

Lemma 6. Fix any $n \geq 2$ and $m \geq 3$. There exists $\epsilon > 0$ such that the equation $MB(\chi(r); r, \kappa) = MC(\chi(r))$ has at most one solution r^* in the range $[\overline{r}_{\text{crit}} - \epsilon, 1]$.

Proof. In the following argument, we defer technical steps to lemmas, which are proved in the Supplementary Appendix.

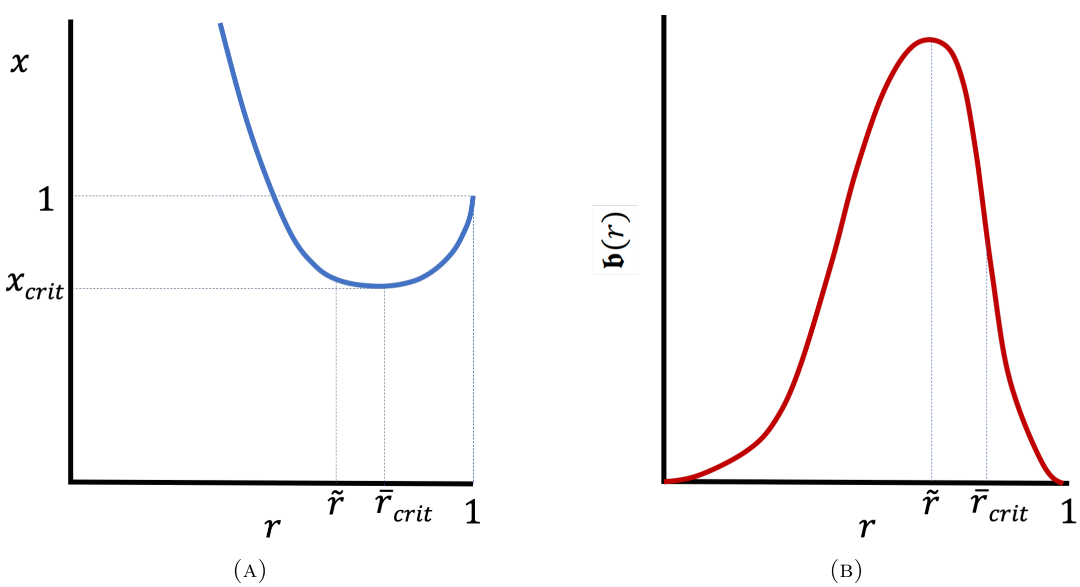


Figure 14. Panel (A) shows the relationship between r and x implied by physical consistency. Panel (B) plots the function $\mathfrak{b}(r)$ discussed in the proof.

Define $\mathfrak{b}(r) = MB(\chi(r); r, \kappa)$ and $\mathfrak{c}(r) = MC(\chi(r))$, so that we can simply study the equation

$$\mathfrak{b}(r) = \mathfrak{c}(r). \tag{16}$$

We can calculate

$$MB(x_{if}; r, \kappa) = \kappa g(r) r n (1 - x_{if} r)^{n-1} m (1 - (1 - x_{if} r)^n)^{m-1}. \tag{17}$$

By plugging in $x = \chi(r)$ into (17), we may rewrite the equation of interest as

$$\underbrace{\kappa g(r) m n r^{2-\frac{1}{m}} \left(1 - r^{1/m}\right)^{1-\frac{1}{n}}}_{\mathfrak{b}(r)} = c'\Bigg(\underbrace{\frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}}_{x} - \underline{x}\Bigg). \tag{18}$$

We wish to show that there is a range $[\overline{r}_{\text{crit}} - \epsilon, 1]$ for which there is at most one solution to equation (16).[53] The right-hand side of the equation is increasing in r for $r \in [\overline{r}_{\text{crit}}, 1]$.[54] If we could establish that the left-hand side, which we call $\mathfrak{b}(r)$, is decreasing in r, it would follow that there is at most a unique r solving (18) for $r \in [\overline{r}_{\text{crit}}, 1]$. Moreover, by the continuity of $\mathfrak{b}(r)$ and $\mathfrak{c}(r)$, it would follow immediately that for some small $\epsilon > 0$, there is also at most a single solution over the range $[\overline{r}_{\text{crit}} - \epsilon, 1]$. Unfortunately, $\mathfrak{b}(r)$ is not decreasing in r. However, we can show that it *is* decreasing in r for $r \geq \overline{r}_{\text{crit}}$, which is sufficient.[55] Panel (B) of Figure 14 gives a representative depiction of $\mathfrak{b}(r)$, reflecting that it is decreasing to the right of $\widetilde{r}$ (which is itself to the left of $\overline{r}_{\text{crit}}$).

The following lemma will, once established, complete the proof.

Lemma 7. $\mathfrak{b}$ is strictly decreasing on the domain $[\overline{r}_{\text{crit}}, 1)$.

To prove Lemma 7 we write $\mathfrak{b}(r)$ as a product of two pieces, $\alpha(r) := \kappa g(r)$ and

$$\beta(r) := mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}}.$$

Note that the function $\beta(r)$ is positive for $r \in (0, 1)$. We will show that it is also strictly decreasing on $[\overline{r}_{\text{crit}}, 1)$. By assumption, $g(r)$ is positive and strictly decreasing in its argument, so $\alpha(r)$ is also positive and decreasing in r. Thus, because $\mathfrak{b}$ is the product of two positive, strictly decreasing functions on $[\overline{r}_{\text{crit}}, 1)$, it is also strictly decreasing on $[\overline{r}_{\text{crit}}, 1)$. It remains only to establish that $\beta(r)$ is strictly decreasing on the relevant domain. Two additional lemmas are helpful.

Lemma 8. The function $\beta(r)$ is quasiconcave and has a maximum at $\widehat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Lemma 9. For all $n \geq 2$ and $m \geq 3$, we have that $\widehat{r} < r_{\text{crit}}$.

Lemmas 8 and 9 are proved in Sections SA1.3 and SA1.4 of the Supplementary Appendix. Together these show that $\beta(r)$ is strictly increasing and then strictly decreasing in r for $r \in (0, 1)$, with a turning point in the interval $(0, \overline{r}_{\text{crit}})$. Thus $\beta(r)$ is strictly decreasing on the domain $[\overline{r}_{\text{crit}}, 1)$, the final piece required to prove Lemma 7.

Thus, by the continuity of $\mathfrak{b}(r)$ and $\mathfrak{c}(r)$, it follows immediately that there exists $\epsilon > 0$ such that there is at most a single solution for $r \in [\overline{r}_{\text{crit}} - \epsilon, 1]$. This completes the proof of the lemma. □

B.4. **Proof of Theorem 1 (Classification of regimes as κ varies).** Having developed the key machinery that we will use to analyze the comparative statics of equilibria, we can proceed to the main proof. We will sometimes take $\underline{x} = 0$ when this is immaterial to the arguments to keep notation uncluttered.

The proof relies mainly on analyzing the shape of the best-response correspondence: how it depends on κ and the reliability level r.

Denote by $BR(r, \kappa)$ the set of values x_{if} maximizing

$$\kappa g(r)\widehat{P}(x_{if}; r) - c(x_{if} - \underline{x}). \tag{19}$$

Recalling the definition (10), this has the interpretation of best-response relationship strengths in the infinite-depth model. Let $\overline{BR}(r, \kappa) = \max\{x : x \in BR(r, \kappa)\}$ be the maximal element of the set $BR(r, \kappa)$. Analogously, let $\overline{BR}(r, \kappa, \mu_\tau) = \max\{x : x \in BR(r, \kappa, \mu_\tau)\}$. This is the finite-$\tau$ analogue of the limit object we have just defined.

Recall the optimal investment first-order condition (OI) in the limit model:

$$MB(x; r, \kappa) = MC(x), \tag{OI}$$

where the definitions are in (11) and (12). We will be using it to study interior optima.

A first lemma in the proof of Theorem 1 is monotonicity of the best response curve in κ.

Lemma 10. $\overline{BR}(r, \kappa)$ is increasing in κ, and strictly so whenever $\overline{BR}(r, \kappa) > 0$.

[53]We note that the functions on both sides of the equation are merely constructs for the proof. In particular, when we sign their derivatives, these derivatives do not have an obvious economic meaning.

[54]This follows directly: χ is increasing on that domain and c' is increasing by assumption. (see Panel (A) of Figure 14).

[55]We do this by showing that the global maximum of $\mathfrak{b}(r)$ is achieved at a number $\widetilde{r}$ that we can prove is smaller than $\overline{r}_{\text{crit}}$.

Proof. Let $x_{if} = \overline{BR}(r, \kappa)$ and suppose OI holds at the optimum. Suppose κ increases slightly. Then, evaluated at x_{if}, the left-hand side of OI increases and the right-hand side does not change as it does not depend on κ. So marginal benefits exceed marginal costs at x_{if}. Now consider increasing x_{if}. By the assumption that $c'(x') \to \infty$ as $x' \to 1$ and the boundedness of marginal benefits, there is an $x' > x$ so that marginal benefits are once again equal to marginal costs. Since the original best-response gave a nonnegative payoff, so must x' (since marginal benefits exceed marginal costs in moving from x to x'). By Assumption 1, the first-order condition is sufficient for a best-response.

On the other hand, if $\overline{BR}(r, \kappa) = 0$ is a strict best response, then by continuity of the value function, it remains so for a slightly higher κ. □

We continue with two lemmas on the best response correspondence. Thereafter, we will prove the three claims of the theorem.

The first lemma concerns the local comparative statics of the intersection between the graph of $r \mapsto \overline{BR}(r, \kappa)$ and the graph of ρ.

Lemma 11. Take any κ_0. Consider any (x_0, r_0) satisfying

$$x = \overline{BR}(r, \kappa) \text{ and } r \in \rho(x) \tag{20}$$

with $x_0 > x_{\text{crit}}$ at $\kappa = \kappa_0$. Then there is a neighborhood around κ_0 where $(X(\kappa), R(\kappa))$ is the unique pair satisfying (20), and both X and R are strictly decreasing, continuous functions of κ.

Proof. Restrict attention to a neighborhood of (x_0, r_0) where $x > x_{\text{crit}}$ and $r > r_{\text{crit}}$. Here ρ is the same as $\widehat{\rho}$. Note that in the notation of Lemma 6, the the conditions (20) are equivalent to

$$\mathfrak{b}(r) = \mathfrak{c}(r) \text{ and } x = \chi(r).$$

The existence of continuous functions $(X(\kappa), R(\kappa))$ extending the solution locally follows by the implicit function theorem applied to this equation. The solution $R(\kappa)$ can be visualized as the horizontal coordinate where the decreasing function $\mathfrak{b}(r)$ in Figure 14(b) intersects the increasing function $c'(\chi(r) - \underline{x})$. As we perturb κ upward, the curve $\mathfrak{b}(r)$ moves up pointwise. Then $R(\kappa)$, the coordinate of the intersection, is increasing in κ. The statement about $X(\kappa)$ follows because X is an increasing function of $R(\kappa)$ in our neighborhood of interest. □

The following lemma concerns the convergence of the maximal best-response function $\overline{BR}(r, \kappa, \mu_\tau)$ in the finite-τ model to the limit model.

Lemma 12. $\overline{BR}(r, \kappa, \mu_\tau)$ converges pointwise to $\overline{BR}(r, \kappa)$ for any r.

Proof of Lemma 12. When it is nonzero, $\overline{BR}(r, \kappa, \mu_\tau) = \max\{x : MB(x; r, \kappa, \mu_\tau) = MC(x)\}$ where

$$MB(x; r, \kappa, \mu_\tau) = \kappa g(r) mn(1 - xr)^{n-1}(1 - (1 - xr)^n)^{m-1} r \sum_{d=1}^{\infty} \mu_\tau(d).$$

Suppose first (passing to a subsequence if necessary) we have a sequence where this these conditions hold.

Note that, since μ_τ puts mass at least $1 - \frac{1}{\tau}$ on $[\tau, \infty)$, it follows that $\sum_{d=1}^{\infty} \mu_\tau(d) \to 1$ as $\tau \to \infty$.

It follows that, as $\tau \to \infty$, we have the following convergence uniformly:

$$MB(x; r, \kappa, \mu_\tau) \to \kappa g(r) mn(1 - xr)^{n-1}(1 - (1 - xr)^n)^{m-1} r = MB(x; r, \kappa)$$

and thus that

$$\overline{BR}(r, \kappa, \mu_\tau) \to \overline{BR}(r, \kappa)$$

for any r.

When $\overline{BR}(r, \kappa, \mu_\tau)$ is identically zero for each τ (again by passing to a subsequence if necessary), it follows from uniform convergence of the firms' value functions to their limit (19) that investing zero strictly dominates any positive level in the limit problem as well. □

We now define some key quantities that will play a role in our proof of the theorem. It is straightforward to see by inspection of the limit maximand (19) and condition (OI) that for any $r \in (0, 1]$ and any $x_0 < 1$ if κ is large enough, all elements of $BR(r, \kappa)$ satisfy $x > x_0$. In particular, we can guarantee that they all lie above x_{crit}. What we have said implies that there is an κ_1 such that for $\kappa > \kappa_1$ the graphs of $\overline{BR}(r, \kappa)$ and the graph of ρ (both viewed as sets of points (x, r)) intersect above x_{crit}. On the other hand, at sufficiently small κ, all elements of $BR(r, \kappa)$ are bounded by a small number. Let $\underline{\kappa} > 0$ be the smallest κ such the

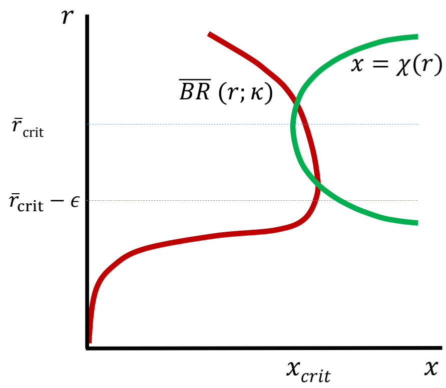


FIGURE 15. The curves $\overline{BR}(r,\kappa)$ and $x = \chi(r)$. If the two were tangent at $r = \overline{r}_{\text{crit}}$ at $\kappa = \underline{\kappa}$, then by perturbing to a slightly higher κ, we would obtain two intersections above $r = \overline{r}_{\text{crit}} - \epsilon$, contradicting Lemma 6.

graph of $\overline{BR}(r,\kappa)$ intersects the graph of ρ at a nonzero value of x. What we have said implies that $\underline{\kappa}$ is a finite, nonzero number.

Consider the two graphs just mentioned at $\kappa = \underline{\kappa}$. Lemma 11 implies that any intersection of the two graphs must be such that $x \leq x_{\text{crit}}$ (and thus $r \leq \overline{r}_{\text{crit}}$), since otherwise we could find an intersection at a smaller value of κ. Thus, $\underline{\kappa}$ is such that $\overline{BR}(r,\underline{\kappa})$ just touches the correspondence $\rho(x)$ at $x = x_{\text{crit}}$. Define $\underline{r}_{\text{crit}}$ to be the point of intersection with the highest r, as depicted in Figure 12(c). By Assumption 1, we may assume that locally, $\overline{BR}(r,\underline{\kappa})$ is exactly the set of solutions to (OI) as a function of r.[56] If it were the case that $\underline{r}_{\text{crit}} = \overline{r}_{\text{crit}}$, then $r \mapsto \overline{BR}(r,\underline{\kappa})$ would be tangent to $r \mapsto \chi(r)$ at $r = \overline{r}_{\text{crit}}$, and by increasing κ slightly we would obtain two intersections between the graphs above $\overline{r}_{\text{crit}} - \epsilon$ a contradiction to Lemma 6. Thus $\underline{r}_{\text{crit}} < \overline{r}_{\text{crit}}$.

We can now prove each part of Theorem 1.

Part (i): For any $\kappa < \underline{\kappa}$, Since $\overline{BR}(0,\kappa) = 0$, the only intersection between $\overline{BR}(r,\kappa)$ and $\rho(x)$ is at $(x,r) = (0,0)$.

Note that for any τ, $\frac{\partial \rho(x,\mu_\tau)}{\partial x_{if}}|_{x_{if}=x=0} = 0$ and $\overline{BR}(0,\kappa) = 0$. Now, since we know $\rho(x,\mu_\tau)$ converges to $\rho(x)$ (Proposition 6) and $\overline{BR}(r,\kappa,\mu_\tau)$ converges to $\overline{BR}(r,\kappa)$ (Lemma 12), it follows that when $\kappa < \underline{\kappa}$, then there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, the curves $\overline{BR}(r,\kappa,\mu_\tau)$ and $\rho(x,\mu_\tau)$ intersect only at a point $(x,r) = (0,0)$.

Part (ii): Now take any $\kappa \geq \underline{\kappa}$. Since $\overline{BR}(r,\kappa) \geq \overline{BR}(r,\underline{\kappa})$ for any r (with equality holding only when $\kappa = \underline{\kappa}$ or possibly when $\overline{BR}(r,\kappa) = 0$), it follows that there is at least one point of intersection between $\overline{BR}(r,\kappa)$ and $\rho(x)$. We select the one with the highest reliability r. If κ is close enough to $\underline{\kappa}$, then this point will be (x_{crit}, r) with $r \in (\underline{r}_{\text{crit}}, \overline{r}_{\text{crit}})$.

Since we know $\rho(x,\mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r,\kappa,\mu_\tau)$ converges to $\overline{BR}(r,\kappa)$, it then follows that for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, $\overline{BR}(r,\kappa,\mu_\tau)$ and $\rho(x,\mu_\tau)$ intersect at some point $x \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$ and $r \in [\underline{r}_{\text{crit}} - \epsilon, \overline{r}_{\text{crit}} + \epsilon]$.

As κ keeps increasing, $\overline{BR}(r,\kappa)$ also increases and thus the point of intersection with the highest reliability will reach $(x_{\text{crit}}, \overline{r}_{\text{crit}})$ when κ reaches some value $\overline{\kappa}$. Since we know $\rho(x,\mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r,\kappa,\mu_\tau)$ converges to $\overline{BR}(r,\kappa)$, by the same argument as before, it then follows that for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, $\overline{BR}(r,\kappa,\mu_\tau)$ and $\rho(x,\mu_\tau)$ intersect at some point $x \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$ and $r \in [\overline{r}_{\text{crit}} - \epsilon, \overline{r}_{\text{crit}} + \epsilon]$.

Part (iii): Finally, as κ increases beyond $\overline{\kappa}$, $\overline{BR}(r,\kappa)$ keeps increasing and the point of intersection between $\overline{BR}(r,\kappa)$ and $\rho(x)$ with the highest reliability r will increase along the part of the $\rho(x)$ curve for which $x > x_{\text{crit}}$, by Lemma 11. At any such point, we also have $r > \overline{r}_{\text{crit}}$.

Since $\rho(x,\mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r,\kappa,\mu_\tau)$ converges to $\overline{BR}(r,\kappa)$, this also holds for any τ large enough. This completes the proof. $\square$

[56]If the best response is indifferent to zero investment, then this indifference can be broken by changing the cost function slightly at inframarginal costs without affecting the equilibrium.

B.4.1. *Proof of Corollary 1 (Comparative statics in baseline institutional quality).* We first consider the limit model. Equilibria are given by the highest intersection of the reliability curve with the best response curve. The reliability curve is constant as both κ and $\underline{x}$ change. The conditions for optimal investment that must be satisfied in the equilibrium $x^{*\prime} := x^*(\kappa', \underline{x})$ are that

$$MB(x^{*\prime}) = \frac{1}{\kappa'} c'(x^{*\prime} - \underline{x}).$$

It needs to be shown that there exists an $\underline{x}' \in (\underline{x}, x^{*\prime})$ such that

$$MB(x^{*\prime}) = \frac{1}{\kappa} c'(x^{*\prime} - \underline{x}').$$

Note that $c'(x^{*\prime} - \underline{x}')$ is continuous in $\underline{x}'$ holding $x^{*\prime}$ fixed, with

$$\frac{1}{\kappa} c'(0) = 0 < \frac{1}{\kappa'} c'(x^{*\prime} - \underline{x}) = MB(x^{*\prime}) \qquad \text{and} \qquad \frac{1}{\kappa} c'(x^{*\prime} - \underline{x}) > \frac{1}{\kappa'} c'(x^{*\prime} - \underline{x}) = MB(x^{*\prime}).$$

The result then follows from the intermediate value theorem.

The extension to the limit as $\tau \to \infty$ is straightforward and analogous to other arguments. □

B.4.2. *Proof of Proposition 3 (Equilibrium fragility).* When $\kappa \in [\underline{\kappa}, \overline{\kappa}]$:

From Theorem 1, for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for any $\tau > \underline{\tau}$, $x^*(\mu_\tau) \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$. Thus for a shock of size 2ϵ, the relationship strength after the shock is $\underline{x} - \epsilon + \xi^*(\mu_\tau) \leq x_{\text{crit}} - \epsilon$.

From Proposition 6, for any $\eta > 0$, there exists $\underline{\tau}'$ such that for all $\tau > \max\{\underline{\tau}, \underline{\tau}'\}$, $\rho(\underline{x} - \epsilon + y^*(\mu_\tau), \mu_\tau) < \eta$.

Thus, when $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ the equilibrium is fragile.

When $\kappa > \overline{\kappa}$:

From Theorem 1, $x^*(\mu_\tau)$ converges to $x^* > x_{\text{crit}}$ as $\tau \to \infty$. Thus, for any $\epsilon > 0$, there may *not* exist a $\underline{\tau}$ such that $x^*(\mu_\tau) - \epsilon = \underline{x} - \epsilon + y^*(\mu_\tau) < x_{\text{crit}}$ for all $\tau > \underline{\tau}$ and from Proposition 6, for any η, there may *not* exist a $\underline{\tau}'$ such that $\rho(\underline{x} - \epsilon + y^*(\mu_\tau), \mu_\tau) < \eta$ for all $\tau > \max\{\underline{\tau}, \underline{\tau}'\}$.

Thus, when $\kappa > \overline{\kappa}$ the equilibrium is robust. □

B.4.3. *Proof of Proposition 4 (Fragility with partial knowledge of depth).* It is helpful to first define and analyze an adjusted baseline model, called the r_A-baseline model. This model is identical to the baseline model except that the reliability of depth 0 firms is replaced by the adjusted reliability $r_A < 1$.

Reliability at depth d in this adjusted baseline model is $\mathcal{R}_x^d(r_A)$ (with $\mathcal{R}^d$ denoting applying the function d times). This is less than reliability in the baseline model at depth d, given by $\mathcal{R}_x^d(1)$ because the function $\mathcal{R}_x$ is increasing. However, if $r_A \geq \overline{r}_{\text{crit}}$ from the baseline model, then in both cases reliability converges from above to $\overline{r}_{\text{crit}}$ as depth increases:

$$\lim_{d\to\infty} \mathcal{R}_x^d(1) = \lim_{d\to\infty} \mathcal{R}_x^d(r_A) = \overline{r}_{\text{crit}}.$$

Thus, if $r_A \geq \overline{r}_{\text{crit}}$, the proof of Theorem 1 and Proposition 3 go through for the r_A-baseline model. In particular, there is equilibrium fragility for the same parameters.

Now consider the case of $r_A < \overline{r}_{\text{crit}}$. We will split the proof into two cases according to the limit value of x. If we have a sequence of equilibria with limit investment $x \leq x_{\text{crit}}$, we can immediately see that the reliability of high-depth firms cannot converge to any $r > \overline{r}_{\text{crit}}$ so the only possibilities are $r = \overline{r}_{\text{crit}}$ or $r = 0$. Thus, any limit equilibrium with investment $x > 0$ must be fragile. Alternatively, if we have a sequence of equilibria with limit investment $x > x_{\text{crit}}$ then in the limit reliability must be greater than r_{crit}. (By Lemmas 3 and 12, the only feasible limit reliability less than r_{crit} is 0 and only the limit investment x_{crit} can yield limit reliability r_{crit}.) Hence

$$\lim_{d\to\infty} \mathcal{R}_x^d(1) = \lim_{d\to\infty} \mathcal{R}_x^d(r_A) > r_{\text{crit}}.$$

As such investments are not a limit equilibrium of the baseline model, they cannot be a limit equilibrium of the r_A-baseline model. Putting these cases together, we conclude that if $x^* > 0$ then the equilibrium must be fragile.

We now argue that in a equilibrium of the partial depth knowledge model the equilibrium behavior of the high-depth firms must be the same as equilibrium behavior in the adjusted baseline model. In a symmetric undominated equilibrium of the partial depth knowledge model the low-depth firms make the same investment. Thus all low-type firms at depth $\overline{d}_l$ have the same reliability. Denote this reliability $\overline{r}$. In such an equilibrium the high-depth firms take this reliability of the low-depth firms as given. Thus the high-depth firm problem is identical to the problem facing firms in the adjusted baseline model, where we set $r_A = \overline{r}$ and let depth d in the adjusted baseline model correspond to depth $\overline{d}_l + d$ for the high-depth firms in the partial depth knowledge model.

We thus conclude, by our analysis of the r_A-adjusted baseline model, that in any symmetric undominated equilibrium of the partial depth knowledge model if there is positive limit investment in equilibrium then there is also equilibrium fragility, as claimed. □

Appendix C. Microfoundations

In this appendix we use a canonical production network model with monopolistic competition to microfound the functionality of firms, firms' profit functions and the planner's objective function—i.e., Properties A, B, B′, and C. Our goal is to keep this part of the model standard, in order to put the focus on the structure of the underlying supply network. The proofs in Appendix B above depend on the structure of production only through the properties just mentioned.

We first study production after investment has occurred and a given realized supply network is in effect; we then discuss the modeling of investment in a prior stage.

C.1. **Production.**

In this subsection we provide a model of production that microfounds the functionality of firms.

C.1.1. *Intermediate and final versions of each variety.* We let production of any variety $v \in \mathcal{V}_i$ be used in one of two ways. First it can be transformed into an intermediate good version, which is usable only by those varieties v' such that $v \in PS_i(v')$—i.e., the varieties v' for which v is a potential supplier. This can be interpreted as a costless transformation made possible by the supply relationship with v' that makes v suitable for use by v'. Alternatively, v can be converted costlessly into a different, consumption good version, denoted $\underline{v}$. (As we will discuss later, this transformation technology is owned by a particular firm, which earns rents from selling this differentiated consumption good.)

C.1.2. *Quantities and production functions.* Suppose v procures for its production $z_{v,v'}$ units of the variety $v' \in \mathrm{S}_j(v)$. For a given required input $j \in I(i)$, let $z_{v,j}$ be the total amount of j sourced by v, summing across all of v's suppliers for this input, and write $\boldsymbol{z}_v$ for the vector of all these quantities associated with variety v. Let ℓ_v be the amount of labor used by variety v. Labor is the only factor (i.e., unproduced good) and it is inelastically supplied, with $\overline{\ell} = 1$ unit of it. The output of v is

$$\phi_v = f(\ell_v, \boldsymbol{z}_v) := (\ell_v)^{\varepsilon_\ell} \prod_{j \in I(i)} (z_{v,j})^{\varepsilon_z}, \tag{21}$$

where $\varepsilon_\ell + |I(i)|\varepsilon_z = 1$ so that there are constant returns to scale. Thus, all varieties in any set $\mathrm{S}_j(v)$ are perfect substitutes. Concerning substitutability *across* varieties, note that production is not possible if one of the inputs cannot be sourced, but on the intensive margin different inputs are substitutable.[57]

Let $q_{\underline{v}}$ be the quantity of $\underline{v}$ consumed. The household consumes aggregate gross production less investments in reliability which are paid for in the aggregated final goods. Aggregate gross production of a *consumption good* is given by

$$Y = \left(\int_{\mathcal{V}} (q_{\underline{v}})^{\eta_C} \, dv \right)^{1/\eta_C}, \tag{22}$$

a demand aggregator with $\frac{1}{2} < \eta_C < 1$.[58] Y_C of the consumption good is allocated to consumption, while Y_I is allocated to investment. Household utility is Y_C. Given investments, efficiency corresponds to maximizing Y.

[57] We view the assumption that all inputs must be sourced for successful production as mild. Indeed, Boehm, Flaaen, and Pandalai-Nayar (2019) estimate, in the context of a supply network disruption event, a production function with low elasticity of substitution, closer to Leontief than Cobb–Douglas.

[58] The same results would hold if we defined a suitable nested-CES aggregator, with categories of products of equal measures being the nests. The upper bound on η_C ensures that the elasticity of substitution between varieties is finite (so that variety is valuable), while the lower bound ensures that the equilibrium returns to increasing variety are concave, which is technically convenient; see the discussion after Definition 1.

C.1.3. *Functionality.* Equation (21) implies that variety $v \in \mathcal{V}_i$ is able to produce if and only if it can procure quantities $(z_{v,v'})_{v,v' \in \mathcal{V}}$ from its suppliers such that $z_{v,j} > 0$ for all $j \in I(i)$. This will be possible if and only if it has at least one link in the realized supply network to a functional supplier for each input it requires—in other words, if and only if it is functional (as defined in Section 2.3). Note that depth-zero nodes have the same production functions as any other nodes; the only difference is that they are not reliant on the operation of specific supply relationships, and always have a functional firm to source their inputs from.

C.2. **Equilibrium on a realized supply network.** We now define a competitive equilibrium and study its structure on a realized supply network. Given a realized supply network, functional firms choose how much of each variety of each input to source (taking prices as given) and what price to set for its consumer good, while the representative consumer supplies a unit of labor inelastically and chooses how much of each consumer good variety to buy (taking prices as given). We start by considering the consumer's problem, and then consider firms' problems before using these to define an equilibrium.

C.2.1. *Equilibrium: Definitions.* Let w denote the wage; p_v the price of variety v when used as an intermediate; and $p_{\underline{v}}$ the price paid by the consumer for the final good corresponding to variety v. The numeraire is the price paid by the consumer for the final output.

As Y_I, the amount of final good devoted to investment, is sunk at the production stage, the household's problem is equivalent to choosing final good consumptions $(q_{\underline{v}})_{\underline{v}}$ to maximize equation (22) subject to the budget constraint

$$\int_{\mathcal{V}} p_{\underline{v}} q_{\underline{v}} d\underline{v} \leq w + \int_{\mathcal{V}} \Pi_v dv, \tag{23}$$

where the right hand side is labor income from the consumer's inelastically supplied unit of labor at wage w and income from the profits Π_v of all firms (including those firms that are not functional). Let $q^*(\underline{p})$ denote the unique input bundle maximizing final good production, where $\underline{p}$ is a vector of the final good prices set by different varieties, and let $q^*_{\underline{v}}(\underline{p})$ denote the corresponding amount of variety v demanded in the production of the final good.

Each firm takes the prices of all intermediate goods as given, as well as the final good prices of other varieties. Prices for intermediate goods are competitive, i.e., equal to marginal costs of production.[59]

Our model is one of monopolistic competition in final goods: a functional firm's problem is to choose input quantities $(z_{v,v'})_{v,v' \in \mathcal{V}}$ from its functional suppliers in the realized supply network, labor demand ℓ_v and a price for its consumer good $\underline{v}$ to maximize its profits.[60] We use efficient pricing for interfirm transactions to focus on inefficiencies coming from network investment, rather than multiple-marginalization issues studied elsewhere, but a model of this form could accommodate wedges in the interfirm prices if desired.

It is convenient to break this problem into two steps. First, for any quantity a firm produces it must choose its inputs to minimize its cost of production. As there are constant returns to scale, this requires choosing inputs to minimize the cost of producing one unit and then scaling these choices up or down to meet the production target. Thus the relative amounts of different inputs and labor used must solve

$$\min_{\boldsymbol{z}_v, \ell_v} \sum_j \sum_{v' \in S_j(v)} p_{v'} z_{v,v'} + w\ell_v \quad \text{subject to } f(\ell_v, \boldsymbol{z}_v) = 1 \tag{24}$$

Let $z^*_{v,v'}$ denote the quantity of variety v' that v chooses to source per unit of its output, and ℓ^*_v the amount of labor that v sources per unit of its output.

Given the unit cost of production generated by solving (24), each variety v sets a price for its consumer good that maximizes its profits (taking reliability investment costs as sunk) and hence solves

$$\max_{p_{\underline{v}}} q^*_{\underline{v}}(\underline{p})[p_{\underline{v}} - \sum_{v'} p_{v'} z^*_{v,v'} - w\ell^*_v] \tag{25}$$

[59]Thus, the intermediate of variety v has a single price irrespective of who buys it. This pricing assumption amounts to requiring efficient production, with no distortions even when a supplier has (in a given realization) market power over a buyer. As documented by (Uzzi, 1997; Kirman and Vriend, 2000), avoiding hold-up is an important function of relational contracts in practice.

[60]As in standard monopolistic competition models, a firm commits to a price $p_{\underline{v}}$ for its consumption good $\underline{v}$ before producing. These goods are sold at a markup above marginal cost; the quantity is determined by consumer demand at that price. As there are constant returns to scale, the amount of intermediate goods produced has no bearing on production costs for the final goods, and vice versa.

Given a realized supply network $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$, with functional firms $\mathcal{V}'$, a competitive equilibrium is given by a specification of intermediate good production $z_{v,v'}$ for all $(v, v') \in \mathcal{E}'$, final good production $q_{\underline{v}}$ for all $v \in \mathcal{V}'$, wage w, intermediate good price p_v for all $v \in \mathcal{V}'$ and final good price $p_{\underline{v}}$ for all $v \in \mathcal{V}'$ such that the following conditions hold:

- the household is choosing how much to consume of each consumer good variety given the prices of these varieties to maximize its utility (22) subject to its budget constraint (23);
- all functional varieties $v \in \mathcal{V}'$ are choosing inputs in ratios that minimize their unit cost given input prices (solve (24));
- all functional varieties $v \in \mathcal{V}'$ are choosing consumer good prices $p_{\underline{v}}$ that maximize their profits given other consumer good prices (solve (25));
- markets clear ($\phi_v = \sum_{v' \in \mathcal{V}'} z_{v,v'} + q^*_{\underline{v}}$ for all $v \in \mathcal{V}'$).

C.2.2. *Equilibrium: Analysis.* Equilibrium is characterized by all firms producing the same quantities of their respective final goods and selling these at the same price. While they affect the propagation of production failures, conditional on being functional local network features do not alter equilibrium prices and quantities. This can be seen by analyzing the prices of different intermediate goods. First, because of constant returns to scale, firms sell their intermediate goods at a price equal to marginal cost. It turns out that within an industry each functional variety is subject to the same intermediate costs, and hence sells at the same intermediate price regardless of their depth. Second, because of the regularity of the supply network, a symmetry exists across industries which results in them selling at the same intermediate prices. These results are shown formally in Lemmas 13 and 14 respectively. In Lemma 15, we show these results imply that all firms sell the same amounts to consumers at the same final prices.

Lemma 13. The selling prices p_v of the intermediate goods of all varieties $v \in \mathcal{V}_i$ within any industry i are equal to some common price p_i.

Proof. As there are constant returns to scale, in equilibrium each intermediate good must be sold at a price equal to their respective marginal costs of production. Thus, by constant returns to scale, the price of each variety's intermediate good, p_v, does not depend on the quantity firm v produces. We establish the conclusion of the lemma by induction on the depth of the variety v. If $d(v) = 0$, firm $v = (i, f)$ can source from any firm in each industry $j \in I(i)$ and can thus minimize its marginal costs. It is easy to deduce from this that $p_v = \min_{f \in [0,1]} p_{(i,f)}$. Now take a variety v of depth $d(v) = d$; by definition v sources its intermediate goods from firms with depth $d - 1$ in each industry j. If these produce at minimal cost so it follows that $p_v = \min_{f \in [0,1]} p_{(i,f)}$. This gives us that all firms in the same industry have the same marginal costs and thus the same intermediate prices, regardless of their depth. $\square$

Lemma 14. For any two industries i and j their intermediate prices are equal: $p_i = p_j$.

Proof. From Lemma 13, there is a price p_i such that all firms in industry i sell intermediate goods at price p_i. By constant returns to scale, p_i is not a function of the quantity produced and is only a function of the prices of the intermediates p_j where $j \in I(i)$. Moreover, each industry requires the same number of inputs and combines these with labor (at common wage w). Thus marginal cost of production for a firm and hence its price can be expressed as a function of just its intermediate good input prices p_j for $j \in I(i)$ and the wage w and by symmetry this is the same function for all industries. We denote this function by $\mathfrak{P}$; it takes as its input m intermediate good prices and the wage. The function is invariant to permutations of intermediate good prices. We can construct a vector of input prices for industry i as $\mathbf{M}_i\mathbf{p}$, where $\mathbf{M}_i$ is a $m \times N$ matrix such that for each row r, the entry $M_{r,k} = 0$ for all k but one which we label $j_i(r)$. This index satisfies $j_i(r) \in I(i)$; $\mathbf{M}_{r,j_i(r)} = 1$; and $\mathbf{M}_{r',j_i(r)} = 0$ for $r' \neq r$. (The matrix simply selects, in each row, one of the inputs of industry i, and the corresponding entry of $\mathbf{M}_i\mathbf{p}$ corresponds to its price.) Then, for all industries i,

$$p_i = \mathfrak{P}(\mathbf{M}_i\mathbf{p}; w),$$

The function $\mathfrak{P}$ is weakly increasing in each input price and strictly increasing in each intermediate input price p_j for $j \in I(i)$.

Now assume, for the sake of contradiction, that not all intermediate input prices are the same. Without loss of generality, let $p_a = \max\{p_1, ..., p_N\}$ and $p_b = \min\{p_1, ..., p_N\}$ with $\frac{p_a}{p_b} = r > 1$. By definition, the price of each intermediate for industry a must be at most p_a, and similarly the price of each input for industry b

must be at least p_b. Thus each element of $\mathbf{M}_a\mathbf{p}$ must be less than or equal to each element of $r\mathbf{M}_b\mathbf{p}$. Now, we claim

$$p_a = \mathfrak{P}(\mathbf{M}_a\mathbf{p}; w) \leq \mathfrak{P}(r\mathbf{M}_b\mathbf{p}; w) < r\mathfrak{P}(\mathbf{M}_b\mathbf{p}; w) = rp_b = p_a.$$

To show the strict inequality, note that strictly increasing all intermediate prices relative to the wage leads to a substitution toward labor, making the increase in unit costs less than proportional. This is a contradiction. Thus, all industries must sell intermediate goods at the same price. □

Lemma 15. For any two functional firms, $v, v' \in \mathcal{V}'$, the prices of these firms' final goods are equal and the same amount of each good is sold: $p_{\underline{v}} = p_{\underline{v}'}$ and $q^*_{\underline{v}}(\underline{p}) = q^*_{\underline{v}'}(\underline{p})$.

Proof. Recall that $q^*_{\underline{v}}(\underline{p})$ denotes the demand of the consumer for final good $\underline{v}$ at prices $\underline{p}$. We first calculate the price elasticity of this demand for final good $\underline{v}$.

The consumers maximize (22) subject to the constraints of (23). Using the method of Lagrange multipliers to solve for the quantity of each variety demanded in terms of prices gives that for any two varieties v, v',

$$\frac{q^*_{\underline{v}}(\underline{p})}{p_{\underline{v}}^{1/(\eta_C-1)}} = \frac{q^*_{\underline{v}'}(\underline{p})}{p_{\underline{v}'}^{1/(\eta_C-1)}} \tag{26}$$

Because there is a continuum of firms, changing the price level and quantity of firm v will not affect the right-hand side of equation (26). Thus, the left-hand side of equation (26) is a constant function of $p_{\underline{v}}$. Upon differentiating this function with respect to $p_{\underline{v}}$, we get

$$\frac{\partial q^*_{\underline{v}}}{\partial p_{\underline{v}}} \cdot p_{\underline{v}} - \frac{q^*_{\underline{v}}}{\eta_C - 1} = 0$$

Rearranging gives us that the price elasticity of demand for variety v at any price level is $\frac{1}{\eta_C-1}$.

Now consider a firm's pricing problem. Given an intermediate good price p (which by Lemmas 13 and 14 is constant for all intermediate goods), there is a unique solution to the cost minimization problem of each firm (Equation (24)).

This marginal cost of production is independent of the quantity produced and is faced by all firms in each industry. Denote this marginal cost c. At a solution to this problem the firm charges a price $p_{\underline{v}}$ that satisfies the Lerner condition:

$$\frac{p_{\underline{v}} - c}{p_{\underline{v}}} = -\frac{1}{\epsilon_d(p_{\underline{v}})}, \tag{27}$$

where ϵ_d is the price elasticity of demand as a function of final good prices. Rearranging equation (27) and substituting the previously calculated elasticity, we have

$$\eta_C p_{\underline{v}} = c \tag{28}$$

It is clear that the left-hand side is injective in $p_{\underline{v}}$ and there is a unique profit-maximizing price that all firms choose. Additionally, as the price elasticity of demand is constant, equation (27) implies that all firms will charge the same constant markup regardless of the quantity of goods they are producing. □

We have shown that in equilibrium: (i) each functional variety v has the same consumer good output $q_{\underline{v}}$, which we will call $\underline{q}$; (ii) these goods are all priced at the same price $p_{\underline{v}}$; (iii) all intermediate goods of variety $v \in \mathcal{V}$ have the same price $p_v = p$. There are two other features of the equilibrium that are crucial:

Lemma 16.

(1) Gross output is equal to $h(\rho(x, \mu))$, where $h : [0, 1] \to \mathbb{R}_+$ is an increasing and continuous function with bounded derivative and $h(0) = 0$.
(2) The expected gross profit of producing each variety, conditional on that variety being functional, is equal to $g(\rho(x, \mu))$; where $g : [0, 1] \to \mathbb{R}_+$ is a decreasing function.

Proof. Note that because labor is the only unproduced input, all labor can be assigned to the production of final goods by allocating labor used to produce intermediate goods to the final good that this intermediate good is ultimately used to produce. Let L_v be the total labor assigned in this way to the production of variety v's final good. As by Lemmas 14 and 15 all firms have equivalent production functions, face the same input prices and choose the same final good prices, they will use the same ratio of different inputs in production. This symmetry throughout the supply network implies that $L_v = L$ for all varieties v. (This does not depend on depth because the zero depth producers have production functions identical to those of other firms and buy inputs—they can just do so from any variety without needing a specific relationship.) As there is a total supply of labor equal to one and it is supplied inelastically, the amount of labor that can be assigned to the production of any given final good is $1/\lambda(\mathcal{V}')$, where $\lambda(\mathcal{V}')$ is the measure of functional firms (and hence equal to $\rho(x,\mu)$ as the measure of firms is 1). As L units of labor are required to produce one unit of any final good, this implies that

$$q^*_{\underline{v}} = q^* = \frac{1}{L\lambda(\mathcal{V}')}. \qquad Y = \left(\lambda(\mathcal{V}')(q^*)^{\eta_C}\right)^{1/\eta_C} = \frac{\lambda(\mathcal{V}')^{1/\eta_C - 1}}{L} = \frac{\rho(x,\mu)^{1/\eta_C - 1}}{L} \tag{29}$$

Setting

$$h(\rho(x,\mu)) = \frac{\rho(x,\mu)^{1/\eta_C - 1}}{L},$$

completes the proof of part (1).

It can be computed that L depends only on the constants in the production functions, and not on x, which establishes (3).[61]

To compute the wage w in terms of the numeraire, we express the aggregate expenditure on final goods in two different ways. On the one hand as the price of the aggregate consumer good has been normalized to 1, it is simply the quantity of aggregate output as given by equation (29) On the other hand, it is the expenditure of the consumer on all firms' final goods, which we simplify using equation (28), and noting that as all inputs are sold at cost all firms have a marginal cost of production equal to wL:

$$p_{\underline{v}} \int_{\mathcal{V}} (q_{\underline{v}}) dv = \rho(x,\mu) \frac{wL}{\eta_C} \frac{1}{\rho(x,\mu)L} = \frac{w}{\eta_C}$$

Combining these two expressions for expenditure implies that

$$w = \left(\frac{\eta_C}{L}\right) \rho(x,\mu)^{\frac{1-\eta_C}{\eta_C}}.$$

From equation (28) each firm sets a final good price of $p_{\underline{v}} = wL/\eta_C$. Thus each firm earns gross profits

$$\left(p_{\underline{v}} - c\right) q^* = w \left(\frac{1-\eta_C}{\eta_C}\right) \frac{1}{\lambda(\mathcal{V}')} = \left(\frac{\eta_C}{L}\right) \rho(x,\mu)^{\frac{1-\eta_C}{\eta_C}} \left(\frac{1-\eta_C}{\eta_C}\right) \frac{1}{\rho(x,\mu)} = \left(\frac{1-\eta_C}{L}\right) \rho(x,\mu)^{\frac{1}{\eta_C} - 2}. \tag{30}$$

This directly implies the second part of the lemma, using the fact that $\eta_C \in (\frac{1}{2}, 1)$. $\square$

Lemma 16 provides microfoundations for Properties A and C. Setting $Y(\mathcal{V}') = h(\rho(x,\mu))$, part (1) shows that maximized aggregate output $Y(\mathcal{V}')$ satisfies Property A. Part (2) shows that the expected gross profit function takes the form $g(r)$ assumed in the model, with $g(r)$ satisfying Property C—i.e., that $g(r)$ is a decreasing, continuously differentiable function. For these statements it is important that L depends only on constants, not on x, which we have shown.

Part (2) of Lemma 16 follows from the household's love of variety. Because of the love of variety that households have, the profit maximizing price that firms set for their consumer goods is at a markup over marginal cost. On the other hand, constant returns to scale in the production function means that intermediate goods that can be produced (i.e., which are able to source all of their required inputs) must be priced at marginal cost in equilibrium. So firms' profits depend just on the markup they charge to consumers, and the quantities they sell to consumers. When more other varieties are functional the consumer's love of variety reduces demand for a given variety and hence the profits obtained by that variety.

C.3. **Endogenizing relationship strengths: Efficient and equilibrium outcomes.** Now we turn to the earlier stage, before production, where firms invest in relationship strengths. Here our goal is to show how the reduced-form modeling of investment costs that we presented in the main text (Section 4.2.2) fits into the general equilibrium model.

[61]Let ℓ be the amount of labor firm v directly hires to produce a unit of its final good. Let z be the amount of each intermediate firm v uses from one of its suppliers to produce a unit of its final good. By symmetry and constant returns to scale, this firm minimizes its unit cost in terms of labor, $L = \frac{\ell}{1-mz}$ subject to the constraint that $1 = z^{m\varepsilon_z}\ell^{\varepsilon_\ell}$. Solving the Lagrangian yields that for some constant γ, we have $\frac{\varepsilon_z}{z} = \frac{\gamma\ell}{(1-mz)^2}$ and $\frac{\varepsilon_\ell}{\ell} = \frac{\gamma}{1-mz}$. Solving this shows that at the optimum $z = \varepsilon_z$, we have $\ell = \varepsilon_z^{-m\varepsilon_z/\varepsilon_\ell}$ and $L = \frac{\varepsilon_z^{-m\varepsilon_z/\varepsilon_\ell}}{\varepsilon_\ell}$.

C.3.1. *The technology of investment.* There is a technology available to each firm to invest the consumption good and increase relationship strengths. Before production, each firm if pledges some amount of the consumption good to invest. We posit that it costs a firm $\frac{1}{\kappa}c(x_{if}-\underline{x})$ units of the numeraire (the consumption good) to achieve relationship strength x with its potential suppliers, where c is a given function—an exogenous technology of investment.

Thus, a firm's expected profit when it is making its investment decision is

$$\Pi_{if} = P(x_{if}; x, \mu)g(r) - \frac{1}{\kappa}c(x_{if} - \underline{x});$$

here both expected revenues and costs are in units of the numeraire. This microfounds the form of the firm profits we have assumed.

C.3.2. *Planner's problem.* We study the problem of efficiently choosing the investments in the first stage. The planner may choose any symmetric investments x for the firms. For a fixed choice of relationship strengths x, let $h(\rho(x, \mu))$ denote the gross production of the aggregated consumption good. As we have said, household consumption is $Y_C = Y - Y_I$, where Y_I is the value of the investment in relationship strength in terms of the quantity of the consumption good devoted to it. These are costs that are sunk prior to production. Obtaining reliability x for all such relationships costs the planner

$$Y_I = \sum_{i \in \mathcal{I}} \int_{v \in \mathcal{V}_i} \frac{1}{\kappa} c(x - \underline{x}) dv = \frac{1}{\kappa} c(x - \underline{x}).$$

(Recall that the measure of the set of all varieties is $\lambda(\mathcal{V}) = 1$.) Setting $c_P(x) = c(x - \underline{x})$ the planner's cost function inherits the key properties of the individual cost function (it is continuous, increasing, and weakly convex with $c(0) = 0$, $c'(0) = 0$ and $\lim_{x \to 1} c(x) = \infty$ by Property B$'$) and hence satisfies the properties assumed in Property B.

Maximizing $Y_C = Y - Y_I$ amounts to our planner's problem from the main text:

$$\max_{x \in [0,1]} h(\rho(x, \mu)) - \frac{1}{\kappa} c_P(x)$$

SUPPLY NETWORK FORMATION AND FRAGILITY
SUPPLEMENTARY APPENDIX FOR ONLINE PUBLICATION

MATTHEW ELLIOTT, BENJAMIN GOLUB, AND MATTHEW V. LEDUC

This document contains supporting material for the paper "Supply Network Formation and Fragility," which herein we refer to as the "main paper" or simply "paper."

SA1. Omitted Proofs

In this section we restate the results in the paper for which proofs were omitted, and then provide the missing proofs.

SA1.1. **Lemma 2.** Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \tag{SA-1}$$

Then there are values $x_{\text{crit}}, \overline{r}_{\text{crit}} \in (0, 1]$ such that:

(i) $\widehat{\rho}(x) = 0$ for all $x < x_{\text{crit}}$;
(ii) $\widehat{\rho}$ has a (unique) point of discontinuity at x_{crit};
(iii) $\widehat{\rho}$ is strictly increasing for $x \geq x_{\text{crit}}$;
(iv) the inverse of $\widehat{\rho}$ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[\overline{r}_{\text{crit}}, 1]$, where $\overline{r}_{\text{crit}} = \widehat{\rho}(x_{\text{crit}})$;
(v) χ is positive and quasiconvex on the domain $(0, 1]$;
(vi) $\chi'(\overline{r}_{\text{crit}}) = 0$.

Proof. We first list some properties of $\widehat{\rho}$ and χ.

Property 0: For positive r in the range of $\widehat{\rho}$, we have $r = \widehat{\rho}(x)$ if and only if

$$x = \frac{1 - (1 - r^{1/m})^{1/n}}{r}.$$

This is shown by rearranging equation (6) in the paper.

Property 1: $\chi(1) = 1$. This follows by inspection.

Property 2: $\chi(r) > 0$ for all $r \in (0, 1]$. This follows by inspection.

Property 3: $\lim_{r \downarrow 0} \chi(r) = \infty$. This follows by an application of l'Hopital's rule, i.e.

$$\lim_{r \downarrow 0} \frac{\frac{d}{dr}\left(1 - (1 - r^{1/m})^{1/n}\right)}{\frac{d}{dr} r} = \lim_{r \downarrow 0} \frac{(1 - r^{1/m})^{1/n - 1} r^{1/m - 1}}{mn} = \infty.$$

Property 4: $\lim_{r \uparrow 1} \chi'(r) = \infty$. This follows by examining

$$\chi'(r) = \frac{r^{1/m}(1 - r^{1/m})^{1/n - 1}}{mnr^2} + \frac{(1 - r^{1/m})^{1/n} - 1}{r^2}$$

Property 5: There is a unique interior $\overline{r}_{\text{crit}} \in (0, 1)$ minimizing $\chi(r)$. To show this, define $z(r) = 1/(1 - r^{1/m})$, and note that $r \in (0, 1)$ satisfies $\chi'(r) = 0$ if and only if the corresponding $z(r) > 1$ solves

$$z - 1 = mn(z^{1/n} - 1);$$

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here we use that the function z is a bijection from $(0,1)$ to $(1,\infty)$. The equation clearly has exactly one solution for $z>1$.[1] Now it remains to see that the unique $r\in(0,1)$ solving $\chi'(r)=0$ defines a local minimum. Note that χ is continuously differentiable on $(0,1)$. Property 3 implies that $\chi'(r)<0$ for some $r<\overline{r}_{\text{crit}}$ while Property 4 implies that $\chi'(r)>0$ for some $r>\overline{r}_{\text{crit}}$. These points suffice to show Property 5.

To prove the lemma, we relate desired properties of $\widehat{\rho}$ to properties of χ; the claims made here can be visualized by referring to Figure 11 in the paper, panels (A) and (B), which illustrate the properties of the functions involved.

Together, Properties 0 and 5 imply that there is no $r>0$ in the range of $\widehat{\rho}$ such that $x<\chi(\overline{r}_{\text{crit}})$. Let $x_{\text{crit}}=\chi(\overline{r}_{\text{crit}})$, which yields (vi) of the lemma; then what we have said implies $\widehat{\rho}(x)=0$ for $x<x_{\text{crit}}$, i.e., statement (i) of the lemma. The proof of Property 5 also implies statement (v) in the Lemma.

It remains to show (ii-iv) of the Lemma. By definition, $\widehat{\rho}(x)$ is the largest solution of (6) in the paper. Properties 1 and 5 imply that on the domain $[\overline{r}_{\text{crit}},1]$, χ is a strictly increasing function whose range is $[x_{\text{crit}},1]$. Fix an $r\in[\overline{r}_{\text{crit}},1]$ and let $x=\chi(r)$. What we have said implies that (x,r) solves (6) in the paper and that there is no r' with $r'>r$ such that (x,r') solves (6) in the paper. Thus, by definition $r=\widehat{\rho}(x)$. Notice that as we vary r in the interval, x varies over the interval $[x_{\text{crit}},1]$. This establishes (iv), and (iii) follows immediately from the fact that χ is increasing on the domain in question. For (ii), it suffices to deduce from Properties 2 and 5 that the minimum of χ has both coordinates positive.

We now consider the case $n=1$. Here, $\frac{d\chi(r)}{dr}=\frac{r^{1/m}-mr^{1/m}}{mr^2}<0$ for all $r\in(0,1]$. Since from Properties 1 and 3 (which still hold when $n=1$), $\chi(1)=1$ and $\lim_{r\downarrow 0}\chi(r)=\infty$, it follows that $\chi(r)$ is decreasing in r and has image $[1,\infty)$. Thus, by logic similar to the above, $\widehat{\rho}(x)=0$ for all $x\in[0,1)$ and $\widehat{\rho}(x)=1$ when $x=1$. It follows that $x_{\text{crit}}=1$ in this case and all the statements of the lemma are satisfied, though some of them are trivial. □

SA1.2. **Lemma 4.** Fix any $m\geq 2$, $n\geq 2$, and $r\geq \underline{r}_{\text{crit}}$. There are uniquely determined real numbers x_1,x_2 (depending on m,n, and x) such $0\leq x_1<x_2<1/r$ and so that:

0. $Q(0;r)=Q(1/r;r)=0$ and $Q(x_{if};r)>0$ for all $x_{if}\in(0,1/r)$;
1. $Q(x_{if};r)$ is increasing and convex in x_{if} on the interval $[0,x_1]$;
2. $Q(x_{if};r)$ is increasing and concave in x_{if} on the interval $(x_1,x_2]$;
3. $Q(x_{if};r)$ is decreasing in x_{if} on the interval $(x_2,1]$.
4. $x_1<x_{\text{crit}}$.

Proof. As a piece of notation, define
$$\zeta(x_{if};r)=1-x_{if}r.$$
When using ζ, we will often omit the arguments for brevity. Then
$$Q(x_{if};r)=mnr\zeta^{n-1}(1-\zeta^n)^{M-1}.$$

Statement 0. It follows immediately from this equation that $Q(0;r)=Q(1/r;r)=0$ and $Q(x_{if};r)>0$ for all $x_{if}\in(0,1/r)$. This establishes Claim 0 in the lemma statement.

Statements 1–3. Establishing Statements 1–3 is more involved; we begin by studying the first derivative of Q to establish the increasing/decreasing statements, and then move to the second derivative to establish the convex/concave statements.

We can calculate
$$Q'(x_{if};r)=-mnr^2\zeta^{n-2}\left(1-\zeta^n\right)^{M-2}\left[\left(mn-1\right)\zeta^n-n+1\right].$$
Note that for $x_{if}\in(0,1/r)$ we have[2]
$$\operatorname{sign}\left[Q'(x_{if};r)\right]=\operatorname{sign}\left[\left(mn-1\right)\zeta^n-n+1\right]. \tag{SA-2}$$
Further, for sufficiently small $x_{if}>0$
$$\operatorname{sign}\left[\left(mn-1\right)\zeta^n-n+1\right]=\operatorname{sign}\left[n(m-1)\right]>0.$$

[1] The left-hand side is linear and the right-hand side is concave, since $n\geq 2$. At $z=1$ the two sides are equal, and the curves defined by the left-hand and right-hand sides are not tangent, so there is exactly one solution $z>1$.

[2] The sign operator is $+1$ for positive numbers, -1 for negative numbers, and 0 when the argument is 0.

Thus $Q'(0;r) > 0$.

We will now deduce from the above calculations about Q' that there is exactly one local maximum of $x_{if} \mapsto Q(x_{if};r)$ on its domain, $[0,1/r]$. First, as this is a continuous function with $Q(0;r) = 0 = Q(1/r;r)$ and $Q'(0;r) > 0$, it follows there is an interior maximum of $Q(x_{if};r)$ in the interval $(0,1/r)$. Next, by (SA-2), $\text{sign}\,[Q'(x_{if};r)] > 0$ if and only if

$$x_{if} < \frac{1 - \left(\frac{n-1}{mn-1}\right)^{\frac{1}{n}}}{r}.$$

Thus, there can be at most one value of x^*_{if} with $Q'(x^*_{if};r) = 0$. Together, these observations imply that Q has one local optimum on its extended domain, which is in fact a global maximum. We let x_2 be defined by the unique value of x_{if} at which $Q'(x^*_{if};r) = 0$. This establishes Property 3. It also establishes the "increasing" part of Properties 1 and 2, since $Q(x_{if};r)$ is increasing to the left of x_2 by what we have said.

The next part of the proof studies the second derivative of Q to establish the claims about the convexity/concavity of Q. First note that

$$Q''(x_{if};r) = mnr^3\zeta^{n-3}\left(1-\zeta^n\right)^{M-3}H$$

where

$$H = \underbrace{\left(m^2n^2 - 3mn + 2\right)}_{A}\zeta^{2n} + \underbrace{\left(\left(1-3m\right)n^2 + \left(3m+3\right)n - 4\right)}_{B}\zeta^n + \underbrace{n^2 - 3n + 2}_{C}.$$

For $x_{if} \in (0,1/r)$, we can see that

$$\text{sign}\left[Q''(x_{if};r)\right] = \text{sign}\left[H\right]. \tag{SA-3}$$

Let $z := \zeta^n$ for $z \in (0,1)$ (which corresponds to $x_{if} \in (0,1/r)$). We can then write $H = \widetilde{H}(z)$ for $z \in (0,1)$, where

$$\widetilde{H}(z) = Az^2 + Bz + C, \tag{SA-4}$$

and A, B and C are constants (labeled above) depending only on m,n. $\widetilde{H}$ is therefore a quadratic polynomial in z and its roots depend only on n and m. Further, $A > 0$, $B < 0$, $C > 0$ and $A+B+C > 0$. Thus $\widetilde{H}$ is convex in z with $\widetilde{H}(0) > 0$ and $\widetilde{H}(1) > 0$.

We first argue that $\min_{z\in[0,1]}\widetilde{H}(z) < 0$. Towards a contradiction suppose $\min_{z\in[0,1]}\widetilde{H}(z) \geq 0$. This implies that Q'' is nonnegative by equation SA-3 and hence that Q is globally convex. However, we have already established that $Q'(0;r) > 0$, so the convexity of Q implies there can be no interior maximum, which contradicts our deductions above.

An immediate implication of $\min_{z\in[0,1]}\widetilde{H}(z) < 0$ is that $\widetilde{H}(z)$ has two real roots, z_1 and $z_2 < z_1$. This establishes the basic shape of $\widetilde{H}(z)$ as illustrated in Figure 1.

It will be helpful to sometimes consider the values of x_{if} that correspond to the roots of the $\widetilde{H}(z)$. To this end, we define the function

$$X(z) := \frac{1 - z^{1/n}}{r}. \tag{SA-5}$$

We can then set $x_1 = X(z_1)$ (i.e., the first inflection point of Q). Along with what we already know, the deduced shape of $\widetilde{H}(z)$ pins down the remaining properties we require about the shape of Q, as we now argue. For an illustration, see Figure 1.

As $z_1 > z_2$, it follows that $\widetilde{H}'(z_1) > 0$. This corresponds to $Q''(x_{if};r)$ going from positive to negative as x_{if} crosses $x_1 := X(z_1)$, and thus $Q(x_{if};r)$ going from convex to concave. As $Q'(0;r) > 0$ and $Q(x_{if};r)$ is convex for $x_{if} \in [0,x_1]$, the maximum of Q must occur at a value of $x_{if} \in (x_1,1)$. Recalling that we let x_2 be the value of x_{if} at which $Q(x_{if};r)$ is maximized we conclude that $x_1 < x_2$. Further, as for values of $x_{if} < x_1$ the function $Q(x_{if};r)$ is convex, we have established the convexity part of Property 2 of Lemma 4.

By similar reasoning, $\widetilde{H}'(z_2) < 0$. This corresponds to $Q''(x_{if};r)$ going from negative to positive as x_{if} crosses $X(z_2)$. Recall that x_2 is defined as the maximum of Q. We must have $X(z_2) > x_2$. If not, Q would be increasing and convex for all $x_{if} \geq X(z_2)$, contradicting the existence of an interior maximum as already established. This establishes that the function Q remains concave until after its maximum

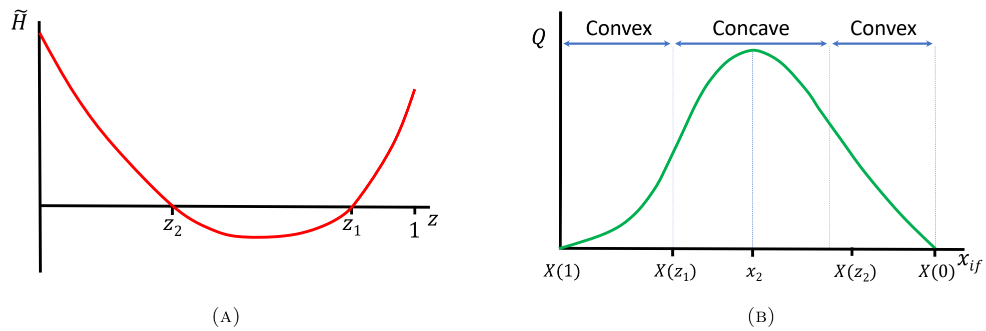


FIGURE 1. Panel (a) shows basic shape of the function $\widetilde{H}(z)$. Panel(b) shows the basic shape of the function $Q(x_{if}; r)$, where the convexity and concavity in different regions is implied by the shape of $\widetilde{H}(z)$.

point x_2 establishing the concavity part of Property 3. This completes our demonstration of Statements 1–3.

Statement 4. We now argue that $x_1 < x_{\text{crit}}$ to establish Statement 4. We do so in two steps. First we show it for the case in which firms other than if have reliability $r = \overline{r}_{\text{crit}}$, and deduce the $r > \overline{r}_{\text{crit}}$ case.

The case $r = \overline{r}_{\text{crit}}$. We need to establish that

$$x_1 = \frac{1 - z_1^{1/n}}{\overline{r}_{\text{crit}}} < \frac{1 - (1 - \overline{r}_{\text{crit}}^{1/m})^{1/n}}{\overline{r}_{\text{crit}}} = x_{\text{crit}}.$$

This holds if and only if $z > 1 - \overline{r}_{\text{crit}}^{1/m}$, which is equivalent to $\overline{r}_{\text{crit}} > (1 - z)^m$. A sufficient condition for this to hold is that $\chi'((1 - z)^m) < 0$, since we know from Lemma 2 in the paper that $\chi'(\overline{r}_{\text{crit}}) = 0$.

Note that

$$\chi'(r) = \frac{\frac{r^{1/m}(1-r^{1/m})^{1/n-1}}{mn} + (1 - r^{1/m})^{1/n} - 1}{r^2}$$

Setting $r = (1 - z)^m$ in the above yields

$$\chi'((1 - z)^m) = \frac{\frac{(1-z)z^{1/n-1}}{mn} + (z)^{1/n} - 1}{(1 - z)^{2m}}.$$

The denominator is always positive, hence we only need to verify that the numerator is negative, when evaluated at the larger of the two roots of $\widetilde{H}$, which we will call z_1. The numerator simplifies to

$$\text{Num}(z) = z^{1/n}\Big(\frac{1 - z}{mnz} + 1\Big) - 1$$

and must now be evaluated at the root z_1. This root is given by the quadratic formula as

$$z_1 = \frac{D + \sqrt{D^2 - 4(n^2 - 3n + 2)(m^2n^2 - 3mn + 2)}}{2(m^2n^2 - 3mn + 2)},$$

where $D = 3mn^2 - 3mn - n^2 - 3n + 4$. This expression for z_1 simplifies to

$$z_1 = \frac{n\Big(3m(n - 1) + \sqrt{(m - 1)(n - 1)(5mn - m - n - 7)} - n - 3\Big) + 4}{2(mn - 2)(mn - 1)}.$$

Now we claim that the shape of Num(z) is pictured in Fig. 2:

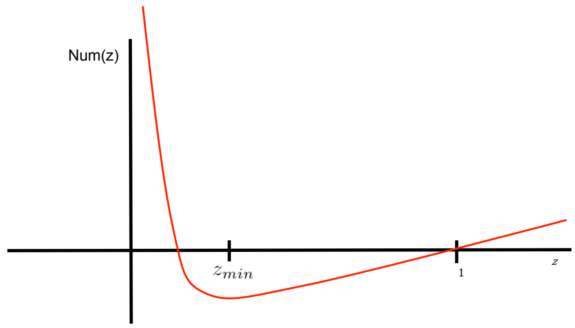


FIGURE 2. Shape of Num(z).

Thus a sufficient condition for $\text{Num}(z_1) < 0$ is that $z_1 > z_{min}$.

We will now calculate $z_{\min}$ and verify the shape of $\text{Num}(z)$ is as claimed. It is easy to show that $z_{min} = \frac{n-1}{mn-1}$ by solving $\frac{d\text{Num}(z)}{dz} = 0$.[3] We also note that since $z^{1/n-2}$ and mn^2 are always positive,

$$\text{sign}\left[\frac{d\text{Num}(z)}{dz}\right] = \text{sign}\left[z(mn-1)+1-n\right],$$

and thus the function $\text{Num}(z)$ is decreasing on $z \in [0, z_{min})$ and increasing on $z \in (z_{min}, \infty)$.

Since $\text{Num}(1) = 0$, the above imply that $\text{Num}(z) < 0$ over the range $[z_{min}, 1)$. Thus, we only need to check that $z_1 > z_{min}$ in order to guarantee that $\text{Num}(z_1) < 0$.

It is easy to verify that the following inequality holds[4]:

$$\begin{aligned} z_{min} &= \frac{n-1}{mn-1} \\ &< \frac{n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\Big) + 4}{2(mn-2)(mn-1)} = z_1. \end{aligned}$$

We conclude (recalling the beginning of this proof) that $z_1 > 1 - \overline{r}_{\text{crit}}^{1/m}$. This establishes that $\widehat{x} < x_{\text{crit}}$ when $r = \overline{r}_{\text{crit}}$.

The case $r > \overline{r}_{\text{crit}}$. From Eq. (SA-5), it is clear that x_1 is decreasing in r. This establishes Property 4 of Lemma 4, completing the proof of Lemma 4. $\square$

SA1.3. **Lemma 8.** The function $\beta(r)$ is quasiconcave and has a maximum at $\widehat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Proof. Recall that $\beta(r) = mnr^{2-\frac{1}{m}}(1-r^{\frac{1}{m}})^{1-\frac{1}{n}}$. We will show that $\beta(r)$ is quasiconcave by demonstrating that there exists an $\widehat{r} \in (0,1)$ such that $\beta'(r) > 0$ for $r \in (0, \widehat{r})$, $\beta'(r) < 0$ for $r \in (\widehat{r}, 1)$ and $\beta'(r) = 0$ for $r = \widehat{r}$.

$$\begin{aligned} \beta'(r) &= mn\left(2 - \frac{1}{m}\right) r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} - \left(1 - \frac{1}{n}\right) mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left(\frac{1}{m} r^{\frac{1}{m}-1}\right) \\ &= mnr^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left[\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right)\frac{1}{m}\right]\right]. \end{aligned}$$

[3]Indeed, $\frac{d\text{Num}(z)}{dz} = \frac{z^{1/n-2}(z(mn-1)+1-n)}{mn^2}$ and thus a first-order condition is satisfied when $z(mn-1)+1-n = 0$.

[4]Indeed, multiplying both sides by $2(mn-2)(mn-1)$ yields

$$2(mn-2)(n-1) < n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\Big) + 4$$

which, after some algebra, reduces to

$$0 < n(mn+1-m-n) + n\sqrt{(m-1)(n-1)(5mn-m-n-7)}.$$

Since all terms on the right-hand side are clearly positive when $m, n \geq 2$, the inequality holds.

Note that the the first factor, $mnr^{1-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{-\frac{1}{n}}$, exceeds 0 for any $r \in (0,1)$ and is equal to 0 for $r = 0, 1$. Moreover, the expression multiplying it,

$$\left(2-\frac{1}{m}\right)-r^{\frac{1}{m}}\left[\left(2-\frac{1}{m}\right)+\left(1-\frac{1}{n}\right)\left(\frac{1}{m}\right)\right],$$

is a strictly decreasing, continuous function of r. It is also strictly positive when $r = 0$ and strictly negative when $r = 1$, implying that it is equal to 0 at some $r \in (0,1)$. It follows that there exists an $\widehat{r} \in (0,1)$ satisfying the claimed properties.

Having established that $\beta(r)$ is quasiconcave and $\beta'(\widehat{r}) = 0$, it follows immediately that $\beta(r)$ is maximized at $r = \widehat{r}$. To find $\widehat{r}$, we use its defining property and solve the following equation:

$$r\left((2m-1)nr^{-1/m}-2mn+1\right)\left(1-r^{1/m}\right)^{-1/n}=0.$$

As $r\left(1-r^{1/m}\right)^{-1/n} > 0$ for any positive production equilibrium, the equation is solved by

$$\widehat{r}=\left(\frac{(2m-1)n}{2mn-1}\right)^{m}.$$

This completes the proof. □

SA1.4. **Lemma 9.** Recall that $\widehat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^{m}$. For all $n \geq 2$ and $m \geq 3$, $\widehat{r} < \overline{r}_{\text{crit}}$.

Proof. By Lemma 2 in the paper, the function $\chi(r)$ is positive and quasiconvex on the domain $[0,1]$ with $\overline{r}_{\text{crit}} = \operatorname{argmin}_r \chi(r)$. Thus, $\widehat{r} < \overline{r}_{\text{crit}}$ if and only if $\chi'(\widehat{r}) < 0$.

Recall that $\chi(r) = \frac{1-(1-r^{1/m})^{1/n}}{r}$ and so

$$\chi'(r)=\frac{\frac{1}{n}(1-r^{1/m})^{1/n-1}\frac{1}{m}r^{1/m}-\left(1-(1-r^{1/m})^{1/n}\right)}{r^2}.$$

We will study this expression evaluated at $\widehat{r} = \left(\frac{(2m-1)n}{2mn-1}\right)^{m}$ from Lemma 8 in the paper.

The denominator of the above expression for $\chi'(r)$ is always positive, so we need only check that the numerator is negative. Calling $A = \frac{n-1}{2mn-1}$ and $B = \frac{n+1-1/m}{n-1}$, we may rewrite the numerator, after some simplifications, as $A^{1/n}B - 1$, and thus we need only check that

$$A^{1/n}B<1 \tag{SA-6}$$

Let $h(n,m) = A^{1/n}B$ .To demonstrate (SA-6), we will show that:

- Step 1: $h(n,3) < 1$, for all $n \geq 2$.
- Step 2: $h(n,m)$ is decreasing in m, for all $n \geq 2$.

Step 1: First note that

$$h(n,3)=\left(\frac{n-1}{6n-1}\right)^{1/n}\left(\frac{n+1-1/3}{n-1}\right)$$

We will show that

- Step 1a: $h(n,3)$ is increasing in n.
- Step 1b: $\lim_{n\to\infty} h(n,3) = 1$.

From this we can conclude that $h(n,3) < 1$, for all $n \geq 2$.

To show Step 1a, note that

$$\frac{\partial h(n,3)}{\partial n}=-\frac{\left(\frac{n-1}{6n-1}\right)^{1/n}\left((18n^2+9n-2)\ln(\frac{n-1}{6n-1})+30n^2+10n\right)}{3n^2\frac{(n-1)}{(6n-1)}}$$

The denominator is positive for all $n \geq 2$. Looking at the numerator, since $\left(\frac{n-1}{6n-1}\right)^{1/n} > 0$ for all $n \geq 2$, it suffices to show that

$$(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right)+30n^2+10n<0 \text{ for all } n\geq 2$$

in order to ensure that $\frac{\partial h(n,3)}{\partial n} > 0$. It is easy to show that

$$\ln\left(\frac{n-1}{6n-1}\right) < \ln\left(\frac{1}{6}\right) < -1.79$$

Thus, for all $n \geq 2$

$$\begin{aligned}(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right)+30n^2+10n &< -(18n^2+9n-2)1.79+30n^2+10n \\ &< -32n^2-16n+4+30n^2+10n \\ &= -2n^2-6n+4 \\ &< 0.\end{aligned}$$

We thus conclude[5] that $h(n,3)$ is increasing in n and Step 1a is proved.

Step 1b follows immediately by noting that

$$\lim_{n\to\infty} h(n,3) = \lim_{n\to\infty}\left(\frac{1}{6}\right)^{1/n} = 1$$

We have thus proved Step 1.

Step 2. To show that $h(n,m)$ is decreasing in m, let us note that, for all $n \geq 2$

$$\begin{aligned}\frac{\partial h(n,m)}{\partial m} &= B\frac{\partial A^{1/n}}{\partial m}+A^{1/n}\frac{\partial B}{\partial m} \\ &= \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-2}{m(n-1)}\frac{mn(1+1/n)-1}{2mn-1}+\frac{1}{(n-1)m^2}\right) \\ &< \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-1}{m(n-1)}+\frac{1}{(n-1)m^2}\right) \\ &< 0.\end{aligned}$$

where the second equality follows after some simplifications, while the first inequality follows from the fact that $\frac{mn(1+1/n)-1}{2mn-1} > \frac{1}{2}$, which is easy to check.

We have thus shown that $h(n,m)$ is decreasing in m, for any $n \geq 2$, and Step 2 is thus proved.

This concludes the proof of the lemma. □

SA2. How production unravels when relationship strength is too low

Figure 3(b) in the main paper shows that when x drops below x_{crit}, the mass of firms that can consistently function falls discontinuously to $\rho(x) = 0$. While we will typically just work with the fixed point as the outcome of interest, the transition will not be instantaneous in practice. How then might the consequences of a shock to x actually play out?

In Figure 3, we work through a toy illustration to shed some light on the dynamics of collapse. Using the same parameters as our previous example, suppose relationship strength starts out at $x = 0.8$. The higher curve in panel (a) is $\mathcal{R}(\cdot; x)$ for this value of x. The reliability of the economy here is r_0, a fixed point of $\mathcal{R}$, which is the mass of functioning firms. Now suppose that a shock occurs, and all relationships become weaker, operating with the lower probability $x = 0.7$. The $\mathcal{R}$ curve now shifts, becoming the lower curve.

To consider the dynamics of how production responds, we must specify a few more details. We sketch one dynamic, and only for the purposes of this subsection. We interpret idiosyncratic link operation realizations as whether a given relationship works in a given period. Before the shift in x, a fraction r_0 of the firms are functional. Let $\widetilde{\mathcal{F}}(0)$ be the random set of functional firms at the time of the shock to x. Now x shifts to 0.7; we can view this as a certain fraction of formerly functional links failing, at random. Then firms begin reacting over a sequence of stages. Let us suppose that at stage s a firm can source its

[5]It is worth noting that this argument does not work for $m = 2$, in which case the numerator of $\frac{\partial h(n,2)}{\partial n}$ could be negative and thus $h(n,2)$ could be decreasing.

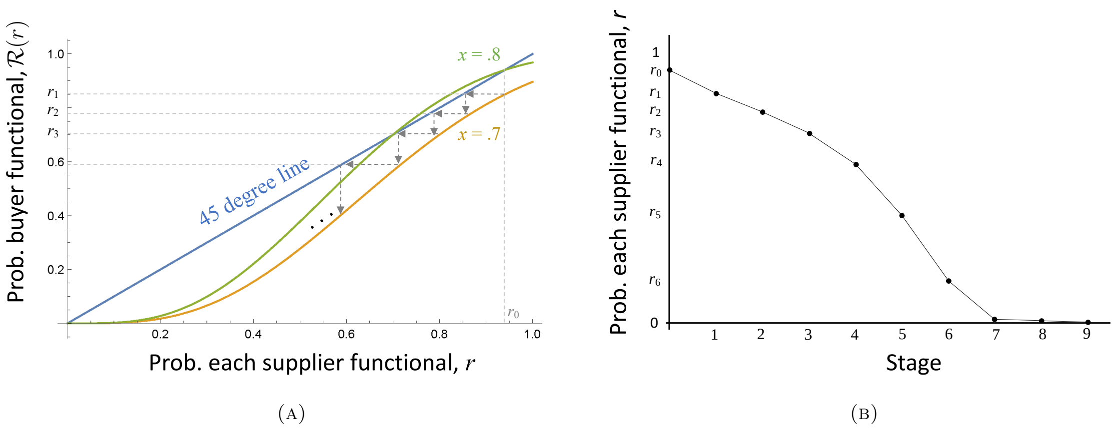


FIGURE 3. The dynamics of unraveling (with the same parameters as in Figure 4 in the main paper, as discussed in Section SA2.

inputs if it has a functional link to a supplier who was functional in the last stage, $s-1$.[6] Let $\widetilde{\mathcal{F}}(s)$ be the set of these functional firms. By the same reasoning as in the previous subsection, we can see that the mass of $\widetilde{\mathcal{F}}(s)$, which we call r_s, is $\mathcal{R}(r_{s-1})$. Iterating the process leads to more and more firms being unable to produce as their suppliers fail to deliver essential inputs. After stage 1, the first set of firms that lost access to an essential input run out of stock and are no longer functional. This creates a new set of firms that cannot access an essential input, and these firms will be unable to produce at the end of the subsequent stage, and so on. The mass of each r_s can be described via the graphical procedure of Figure 3: take steps between the $\mathcal{R}$ curve and the 45-degree line.

This discussion helps make three related points. First, even though the disappearance of the positive fixed point—and thus the possibility of a positive mass of consistently functional firms—is sudden, the implications can play out slowly under natural dynamics.[7] The first few steps may look like a few firms being unable to produce, rather than a sudden and total collapse of output.

The second point is more subtle. Suppose that when the dynamic of the previous paragraph reaches r_2, the shock is reversed, x again becomes 0.8, and $\mathcal{R}$ again becomes the higher curve. Then, with some supply links reactivated, some of the firms that were made non-functional as the supply chain unravelled will become functional again, and this will allow more firms to become functional, and so on. Such dynamics could take the system back to the r_0 fixed point if sufficiently many firms remain functional at the time the shock is reversed. Thus, our theory predicts that *sufficiently persistent* shocks to relationship strength lead to eventual collapses of production, but, depending on the dynamics, the system may also be able to recover from sufficiently transient shocks.

The third point builds on the second. Suppose that a shock is anticipated and expected to be temporary. Then firms may take actions that slow the unravelling to reduce their amount of downtime. For example, they may build up stockpiles of essential inputs. If all firms behave in this way, the dynamics can be substantially slowed down and the possibility of recovery will improve.

Having illustrated some of the basic forces and timing involved in unraveling, we do not pursue here a more complete study of the dynamics of transient shocks, endogenous responses, etc.—an interesting subject in its own right. Instead, from now on we will focus on the size, $\rho(x)$, of the consistent functional set, which is the steady-state outcome under a relationship strength x.

[6]For example, firms might hold inventories that enable them to maintain production for a certain amount of time, even when unable to source an essential input. Even if firms engage in just-in-time production and do not maintain inventories of essential inputs, there can be a lag between shipments being sent and arriving.

[7]Indeed, in a more realistic dynamic, link realizations might be revised asynchronously, in continuous time, and firms would stop operating at a random time when they can no longer go without the supplier (e.g., when inventory runs out). Then the dynamics would play out "smoothly," characterized by differential equations rather than discrete iteration.

SA3. Interdependent supply networks and cascading failures

We now posit an interdependence among supply networks wherein each firm's profit depends on the *aggregate* level of output in the economy, in addition to the functionality of the suppliers with whom it has supply relationships. Formally, suppose, $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$, where $K_{\mathfrak{s}}$ is a strictly increasing function and Y is the integral across all sectors of equilibrium output:

$$Y = \int_{\mathcal{S}} \rho(x^*_{\mathfrak{s}}) d\Phi(\mathfrak{s}).$$

Here we denote by $x^*_{\mathfrak{s}}$ the unique positive equilibrium in sector $\mathfrak{s}$. The output in the sector is the reliability in that sector, $\rho(x^*_{\mathfrak{s}})$.

The interpretation of this is as follows: When a firm depends on a different sector, a specific supply relationship is not required, so the idiosyncratic failure of a given producer in the different sector does not matter—a substitute product can be readily purchased via the market. Indeed, it is precisely when substitute products are not readily available that the supply relationships we model are important. However, if some sectors experience a sudden drop in output, then other sectors suffer. They will not be able to purchase inputs, via the market, from these sectors in the same quantities or at the same prices. For example, if financial markets collapse, then the productivity of many real businesses that rely on these markets for credit are likely to see their effective productivity fall. In these situations, dependencies will result in changes to other sectors' profits even if purchases are made via the market. Our specification above takes interdependencies to be highly symmetric, so that only aggregate output matters, but in general these interdependencies would correspond to the structure of an intersectoral input-output matrix, and K would be a function of sector level outputs, indexed by the identity of the sourcing sector.

This natural interdependence can have very stark consequences. Consider an economy characterized by a distribution Ψ in which the subset of sectors with $m \geq 2$ has positive measure, and some of these have positive equilibria. Suppose that there is a small shock to $\underline{x}$. As already argued, this will directly cause a positive measure of sectors to fail. The failure of the fragile sectors will cause a reduction in aggregate output. Thus $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$ will decrease in other sectors *discontinuously.* This will take some other sectors out of the robust regime. Note that this occurs due to the other supply chains failing and not due to the shock itself. As these sectors are no longer robust, they topple too following an infinitesimal shock to $\underline{x}$. Continuing this logic, there will be a domino effect that propagates the initial shock. This domino effect could die out quickly, but need not. A full study of such domino effects is well beyond our scope, but the forces in the very simple sketch we have presented would carry over to more realistic heterogeneous interdependencies.

Fig. 4 shows[8] how an economy with 100 interdependent sectors responds to small shocks to $\underline{x}$. In this example, sectors differ only in their initial κ's. The technological complexity is set to $m = 5$ and the number of potential suppliers for each firm is set to $n = 3$. The cost function[9] for any firm if is $c(x_{if} - \underline{x}) = \frac{0.01}{(1-(x_{if}-\underline{x}))^2}$ while the gross profit function is $g(\rho(x)) = 5(1-\rho(x))$. This setup yields values $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$ delimiting the region corresponding to critical (and therefore fragile) equilibria, as per Theorem 1.

In Fig. 4(a), the productivity shifter of a given sector is distributed uniformly, i.e. $\kappa_{\mathfrak{s}} \sim U(\underline{\kappa}, 25)$, so that many sectors have high enough productivity to be in a robust equilibrium while a small fraction have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors thus causes the failure of the fragile sectors (there are initially 12 of them). This then decreases the output Y across the whole economy, but only to a small extent (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is thus only enough to bring one robust sector into the fragile

[8]Note that in this example, the cascade dynamics is as follows: At step 1, firms in sectors with a κ in the fragile range fail due to an infinitesimal shock to $\underline{x}$. The initial economy-wide output Y_1 is then decreased to Y_2 and the κ's are updated using an updating function $K(Y)$ increasing in Y. Only then, are the firms in the surviving sectors allowed re-adjust x_{if}. At step 2, infinitesimal shocks hit again and the firms newly found in the fragile regime fail. This process goes on at each step until no further firm fails, at which point the cascade of failures stops.

[9]For simplicity, we set $\underline{x} = 0$. An infinitesimal shock to $\underline{x}$ has the effect of causing the firms of sectors in the fragile regime to fail, but does not affect the value of $\underline{x}$, which remains at 0.

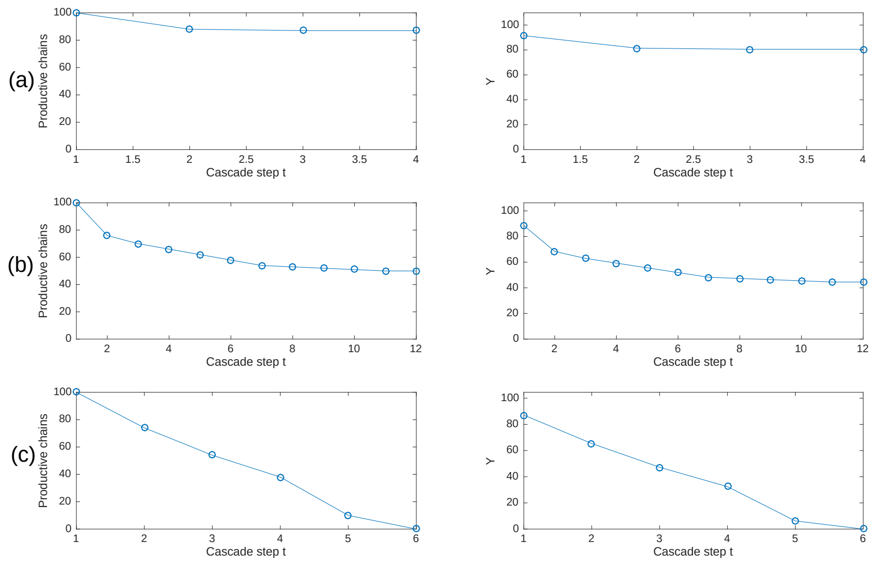


FIGURE 4. Number of sectors that remain productive (left) and economy-wide output Y (right) for each step of a cascade of failures among 100 interdependent sectors. For all sectors: $n = 3$, $m = 5$, $c(x_{if} - \underline{x}) = \frac{0.01}{(1-(x_{if}-\underline{x}))^2}$, $g(\rho(x)) = 5(1 - \rho(x))$. This yields $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$. In row (a), 100 sectors have $\kappa_{\mathfrak{s}}$ initially distributed according to $U(\underline{\kappa}, 25)$; In row (b), 100 sectors have $\kappa_{\mathfrak{s}}$ initially distributed according to $U(\underline{\kappa}, 13)$; In row (c), 100 sectors have $\kappa_{\mathfrak{s}}$ initially distributed according to $U(\underline{\kappa}, 10)$.

regime and thus to cause it to fail as well upon a small shock. In the end, a total of only 13 sectors have failed.

In contrast, Fig. 4(b) shows an economy where $\kappa_{\mathfrak{s}} \sim U(\underline{\kappa}, 13)$, so that more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors causes the failure of the fragile sectors (now initially 24). These have a larger effect on decreasing the output Y across the whole economy (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is now enough to bring many of them into the fragile regime and to cause them to fail upon an infinitesimal shock to $\underline{x}$. This initiates a cascade of sector failures, ultimately resulting in 50 sectors ceasing production.

Fig. 4(c) shows an economy where $\kappa_{\mathfrak{s}} \sim U(\underline{\kappa}, 10)$, so that even more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors causes the failure of the fragile sectors (now initially 26) and this initiates a cascade of sector failures which ultimately brings down all 100 sectors of the economy.

The discontinuous drops in output caused by fragility, combined with the simple macroeconomic interdependence that we have outlined, come together to form an amplification channel reminiscent, e.g., of Elliott, Golub, and Jackson (2014) and Baqaee (2018). Thus, the implications of those studies apply here: both the cautions regarding the potential severity of knock-on effects, as well as the importance of preventing first failures before they can cascade.

SA4. Production networks with limited depth

The main results of the paper are stated for supply chain depth d sufficiently large. In this section, we numerically explore the shape of the reliability function $\tilde{\rho}(x, d)$ at realistic values of d. We do so while allowing for some systematic heterogeneity (for example, more upstream tiers being simpler).

To motivate the literal modeling of assembly in finitely many tiers, we return to the example of an Airbus A380. This product has 4 million parts. The final assembly in Toulouse, France, consists of six large components coming from five different factories across Europe: three fuselage sections, two wings, and the horizontal tailplane. Each of these factories gets parts from about 1500 companies located in 30 countries[10]. Each of those companies itself has multiple suppliers, as well as contracts to supply and maintain specialized factory equipment, etc.

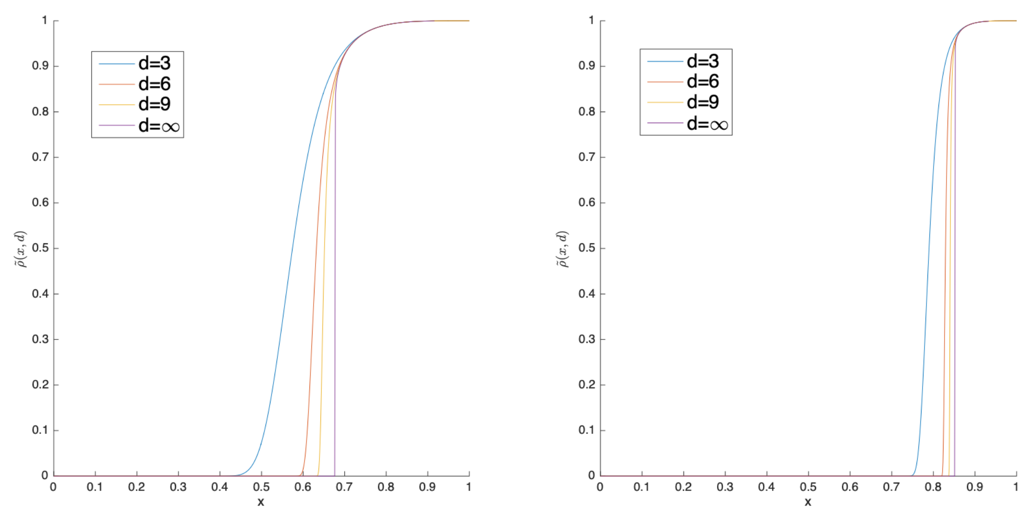
As in the main paper consider a depth-d supply tree, but let each firm in tier $t \in \{0, 1, ..., d\}$ require m_t kinds of inputs and have n_t potential suppliers of each input. Here $t = d$ is the most downstream tier and $t = 0$ is the most upstream tier. The nodes at tier $t = 0$ are functional for sure. As before, we denote by $\widetilde{\rho}(x, d)$ the probability of successful production at the most downstream node of a depth-d tree with these properties. This is defined as

$$\widetilde{\rho}(x, d) = (1 - (1 - \widetilde{\rho}(x, d - 1)x)^{n_d})^{m_d}$$

with $\widetilde{\rho}(x, 0) = 1$, since the bottom-tier nodes do not need to obtain inputs.

We see that the expression is recursive and, if unraveled explicitly, would be unwieldy after a number of tiers. However, we know from Definition 5 and Proposition 6 (both in the paper) that when $m_t = m$ and $n_t = n$ for all t, then for any $x \in [0, 1]$, as d goes to infinity, $\widetilde{\rho}(x, d)$ converges to the correspondence $\rho(x)$.

We start with some examples where m_t and n_t are the same throughout the tree. Figure 5 illustrates the successful production probability $\widetilde{\rho}(x, d)$ for different finite depths d and how quickly it converges to the correspondence $\rho(x)$.



(A) (B)

FIGURE 5. Successful production probability $\widetilde{\rho}(x, d)$ for different finite numbers of tiers d. In panel (A), $m = 5$ and $n = 4$. In panel (B), $m = 40$ and $n = 4$.

In panel (A), we see that $\widetilde{\rho}(x, d)$ exhibits a sharp transition for a depth as small as $d = 3$. The red curve ($d = 6$) shows that when the investment level x drops from 0.66 to 0.61, or about 7 percent, production probability $\widetilde{\rho}(x, 6)$ drops from 0.8 to 0.1. (Thus $\widetilde{\rho}(x, 6)$ achieves a slope of at least 14.) In panel (B), we see that increasing product complexity (by increasing m to 40) causes $\widetilde{\rho}(x, d)$ to lie quite close to $\rho(x)$. This illustrates how complementarities between inputs play a key role in driving this sharp transition in the probability of successful production. Note that $m = 40$ is not an exaggerated number in reality. In the Airbus example described earlier, many components would exhibit such a level of complexity.

However, a complexity number like $m = 40$ will not occur everywhere throughout the supply network. Indeed, and more generally, one might ask whether the regularity in the production tree is responsible for the sharpness of the transition. To investigate this possibility, in Figure 6 we plot $\widetilde{\rho}(x, d)$ for a supply

[10]Source: "FOCUS: The extraordinary A380 supply chain". *Logistics Middle East*. Retrieved on 28 may, 2019 from `https://www.logisticsmiddleeast.com/article-13803-focus-the-extraordinary-a380-supply-chain`

tree with irregular complexity, where different tiers may have different values of m_t. Here we construct 4 trees whose complexity increases with d. The first trees has $d = 3$ and $m_1 = 2$, $m_2 = 6$ and $m_3 = 10$. The second tree has $d = 6$, $m_t = 2$ for $t = 1, 2$, $m_t = 6$, for $t = 3, 4$ and $m_t = 10$, for $t = 5, 6$. The third tree has $d = 9$, $m_t = 2$ for $t = 1, 2, 3$, $m_t = 6$, for $t = 4, 5, 6$ and $m_t = 10$, for $t = 7, 8, 9$. Finally, the fourth tree has large depth (here $d = 999$), $m_t = 2$ for t below the first tercile, $m_t = 6$ for t between the first and the second terciles and $m_t = 10$ for t above the second tercile. We see that trees of moderate depth once again exhibit a sharp transition in their probability of successful production. This feature is thus not at all dependent upon the regularity of the trees.

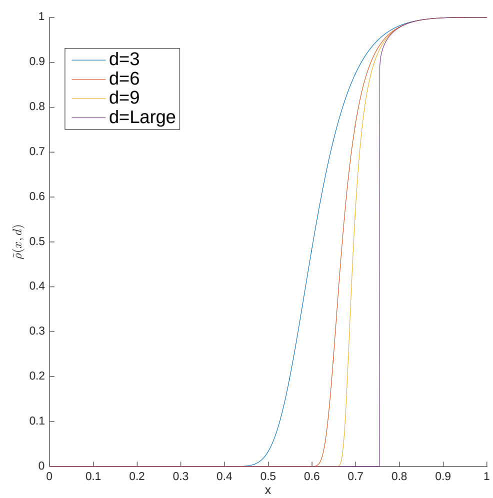


FIGURE 6. Successful production probability $\widetilde{\rho}(x, d)$ for different finite numbers of tiers d, but where different tiers may have different complexity m.

SA5. HETEROGENEITY

There is a finite set of products, $\mathcal{I}$. Each product $i \in \mathcal{I}$ is associated with a *product complexity* m_i and a finite set of inputs $I(i) \subseteq \mathcal{I}$ of cardinality m_i. Thus, the number of inputs required can be different for different products. For each product i and input $j \in I(i)$, there is a number n_{ij} of potential suppliers of product j that each firm has; thus n_{ij} replaces the single multisourcing parameter n. For each pair $i, j \in \mathcal{I}$, there is a relationship strength x_{ij} such that every link where product i sources product j has a probability x_{ij} of being operational.

We generalize our basic model of depth by assigning each firm a generalized depth. Let the set of depth types be $\mathcal{D}$, indexed by the nonnegative integers, with a typical element denoted δ. The lowest depth type firms, with $\delta = 0$, require no specific sourcing. Any higher depth type firm sources only from depth types lower than it. We then construct the potential supply network as follows: Each firm in industry i has a depth type $\delta_{if} \in \mathcal{D}$. There is a distribution $\mu_i \in \Delta(\mathcal{D})$ of depth types in product i. For every $i \in \mathcal{I}$, $j \in I(i)$, and $\delta \in D$ with $\delta > 0$ and $\mu_i(\delta) > 0$, there is a measure $\mu_{ij,\delta} \in \Delta(\mathcal{D} \cap [0, \delta))$ which is the distribution of depths of product j when i sources from j, assuming $n_{ij} > 0$. The distribution of the depths of if's suppliers can be different in different inputs j. A depth type has no required inputs if and only if it is of depth zero. We assume that draws of $\delta_{jf'}$ are independent across different suppliers jf' (just for simplicity). We assume that if there is a positive measure of firms of depth δ in product i and some j, δ' for which $\mu_{ij,\delta}(\delta') > 0$ (i.e., there is a positive probability of depth-δ firms in i matching with depth-δ' firms in j) then $\mu_j(\delta') > 0$.[11]

[11]We do not need to impose any other accounting identities, since the average number of in-links to any variety is arbitrary and not does not play a role in our results. An equivalent way of looking at the structure just described is that supply networks are (as before) directed acyclic graphs—see Tintelnot et al. (2018) for a similar model. This implies that there are some firms that require no specific sourcing. We assign these firms depth $\delta = 0$. Removing these firms from the network, there must then be some remaining firms that require no specific sourcing from the other remaining firms. These are depth

We also need to impose conditions to make the finite depth model converge in a suitable sense to an infinite model. Define a profile

$$\boldsymbol{\mu}(\tau) = [(\mu_i(\tau))_{i\in\mathcal{I}}, (\mu_{ij,\delta}(\tau))_{i\in\mathcal{I}, j\in I(j), \delta\in\mathcal{D}}]$$

of depth distributions to be parameterized by τ as in the basic model. We impose a condition that captures that, as $\tau \to \infty$, chains become deeper.

Assumption SA1.

(1) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\delta, \delta' \in D$ such that $\delta' > \delta'$, the distribution $\mu_{ij,\delta'}$ first-order stochastically dominates $\mu_{ij,\delta}$.
(2) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\overline{\delta} \in D$, the mass placed on $[0, \overline{\delta}]$ by $\mu_{ij,\delta}(\tau)$ tends to 0 as $\delta \to \infty$.
(3) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\overline{\delta} \in D$, the mass placed on $[0, \overline{\delta}]$ by $\mu_i(\tau)$ tends to 0 as $\tau \to \infty$.

Now we can iteratively define reliabilities for this model. Let $\widetilde{\rho}_{i,\delta}(x_{ij}) = 1$ if $\delta = 0$ and, inductively, for $\delta > 0$,

$$\widetilde{\rho}_i(\delta) = \prod_{j\in I(i)} \mathbb{E}_{\delta'\sim\mu_{ij,\delta}} \left[1 - (1 - \widetilde{\rho}_j(\delta')\mathrm{x}_{ij})^{n_{ij}}\right]. \tag{SA-7}$$

In the expectation, we are summing over different values of δ', weighted by probabilities $\mu_{ij,\delta}(\delta')$. Analogously to the main model, we define

$$\rho_i = \mathbb{E}_{\delta\sim\mu_{ij}} \widetilde{\rho}_i(\delta), \tag{SA-8}$$

and we let $\boldsymbol{\rho}(\boldsymbol{x}, \boldsymbol{\mu})$ denote the (multidimensional) reliability function thus constructed.

Again paralleling the main model, we will now study a function that corresponds to the limit of the system (SA-7) as $\tau \to \infty$. We will use the notation r_i for the fraction of firms in product i functioning, and let $\boldsymbol{r}$ denote a vector of all such reliabilities. Define $\mathcal{R}(\boldsymbol{r})$ by

$$[\mathcal{R}(\boldsymbol{r})]_i = \prod_{j\in I(i)} \left[1 - (1 - r_j\mathrm{x}_{ij}(\xi))^{n_{ij}}\right]. \tag{SA-9}$$

Lemma **SA1.** There is a pointwise-largest fixed-point of the function $\mathcal{R}(\boldsymbol{r})$, denoted by $\widehat{\boldsymbol{\rho}}(x)$. Moreover, $\boldsymbol{\rho}(x, \tau)$ converges pointwise to $\widehat{\boldsymbol{\rho}}(x)$ as $\tau \to \infty$.

Proof. By Tarski's theorem, since $\mathcal{R}$ is monotone, it has a pointwise largest fixed point.

By inductively using Assumption SA1(1), it can be shown that $\widetilde{\boldsymbol{\rho}}(\delta)$ is pointwise decreasing in δ. This monotone sequence must have a limit point, and by applying Assumption SA1(2) we can see that it must be a fixed point of $\mathcal{R}$. Moreover, the argument of Echenique (2005, Section 4) shows that it is the largest fixed point. Finally, Assumption SA1(3) guarantees that $\boldsymbol{\rho}(x)$ converges to this fixed point as well. □

We now prove that a sharp transition in production probability arises in the heterogeneous setting (Proposition SA1) by analyzing the $\widehat{\boldsymbol{\rho}}(x)$ defined in Lemma SA1. We then present two examples exhibiting the fragility and weakest link properties (Subsection SA5.3). Finally, we present numerical examples and show that fragility is compatible with endogenous investment (Subsection SA5.3.5).

SA5.1. **Generalization of the sharp transition in the heterogeneous case.** To show the existence of a discontinuity in $\widehat{\boldsymbol{\rho}}$, we study the function along a single curve in $\boldsymbol{x}$ space. To this end we introduce a single parameter ξ and define $x_{if,j} = \mathrm{x}_{ij}(\xi)$, where $\mathrm{x_{ij}}$ are strictly increasing, differentiable, surjective functions $[0,1] \to [0,1]$ with $\mathrm{x}_{ij}(1) = 1$.

Proposition **SA1.** Suppose that for all products i, the complexity m_i is at least 2. Moreover, suppose whenever $j \in I(i)$, the number n_{ij} of potential suppliers for each firm is at least 1. For any product i, the measure of the set of functional firms $\overline{\mathcal{F}}_i$, denoted by $\rho_i(\xi)$, is a nondecreasing function with the following properties.

$\delta = 1$ firms. Proceeding iteratively we assign all firms a depth. This allow a firm to source from other firms with different depths but implies that a depth k firm source only from firms with depth less than k.

(1) There is a number ξ_{crit} and a vector $\boldsymbol{r}_{\text{crit}} > 0$ such that $\boldsymbol{\rho}$ has a discontinuity at ξ_{crit}, where it jumps from 0 to $\boldsymbol{r}_{\text{crit}}$ and is strictly increasing in each component after that.
(2) If $n_{ij} = 1$ for all i and j, we have that $\xi_{\text{crit}} = 1$.
(3) If $\xi_{\text{crit}} < 1$, then as ξ approaches ξ_{crit} from above, the derivative $\rho_i'(x)$ tends to ∞ in some component.

The idea of this result is simple, and generalizes the graphical intuition of Figure 4 in Section 3.1.1 of the paper. For any $\boldsymbol{r} \in [0,1]^{|\mathcal{I}|}$, define $\mathcal{R}_\xi(\boldsymbol{r})$ to be the probability, under the parameter ξ, that a producer of product i is functional given that the reliability vector for producers of other products is given by $\boldsymbol{r}$. This can be written, using (SA-9), as

$$[\mathcal{R}_\xi(\boldsymbol{r})]_i = \prod_{j \in I(i)} \left[1 - (1 - r_j \mathrm{x}_{ij}(\xi))^{n_{ij}}\right].$$

Near 0, the map $\mathcal{R}_\xi : [0,1]^n \to [0,1]^n$ is bounded above by a quadratic function (as a consequence of $m_i \geq 2$ for all i). Therefore it cannot have any fixed points near 0. Thus, analogously to Figure 4 in the main paper, fixed points disappear abruptly as ξ is reduced past a critical value ξ_{crit}.

Proof of Proposition SA1. For any $\boldsymbol{r} \in [0,1]^{|\mathcal{I}|}$, define $\mathcal{R}_\xi(\boldsymbol{r})$ to be the probability, under the parameter ξ, that a producer of product i is functional given that the reliability vector for all products is given by $\boldsymbol{r}$. This can be written explicitly:

$$[\mathcal{R}_\xi(\boldsymbol{r})]_i = \prod_{j \in I(i)} \left[1 - (1 - r_j \mathrm{x}_{\mathrm{ij}}(\xi))^{n_{ij}}\right].$$

Let $\widehat{\boldsymbol{\rho}}(\xi)$ be the elementwise largest fixed point of $\mathcal{R}_\xi$, which exists and corresponds to the mass of functional firms by the same argument as in Lemma 2 in the main paper.

(1) It is clear that $\widehat{\boldsymbol{\rho}}(1) = \mathbf{1}$.

(2) Next, there is an $\epsilon > 0$ such that if $\|\boldsymbol{r}\| < \epsilon$, then for all ξ, the function $\mathcal{R}_\xi(\boldsymbol{r}) < \boldsymbol{r}$ elementwise. So there are no fixed points near $\mathbf{0}$.

(3) For small enough ξ, the function $\mathcal{R}_\xi$ is uniformly small, so $\widehat{\boldsymbol{\rho}}(\xi) = \mathbf{0}$.

These facts together imply that $\widehat{\boldsymbol{\rho}}$ has a discontinuity where it jumps up from $\mathbf{0}$. Let ξ_{crit} be the infimum of the ξ where $\widehat{\boldsymbol{\rho}}(\xi) \neq 0$.

Define

$$\Gamma(\mathcal{R}_\xi(\boldsymbol{r})) = \{(\boldsymbol{r}, \mathcal{R}_\xi(\boldsymbol{r})) : \boldsymbol{r} \in [0,1]^{|\mathcal{I}|}\}$$

to be the graph of the function. What we have just said corresponds to the fact that this graph intersects the diagonal when $\xi = \xi_{\text{crit}}$, but not for values $\xi < \xi_{\text{crit}}$. Suppose now, toward a contradiction, that the derivative $\boldsymbol{\delta}(\xi)$ of $\widehat{\boldsymbol{\rho}}(\xi)$ is bounded in every coordinate as $\xi \downarrow \xi_{\text{crit}}$. Then, passing to a convergent subsequence and using the smoothness of $\mathcal{R}$, we find that the derivative $\boldsymbol{\delta}(\xi_{\text{crit}})$ of $\widehat{\boldsymbol{\rho}}$ is well-defined at $\xi = \xi_{\text{crit}}$. But that contradicts our earlier deduction that $\widehat{\boldsymbol{\rho}}$ is discontinuous at ξ_{crit}. □

This result shows that there are profiles $\boldsymbol{x}$ of relationship strengths where small reductions in strength cause a discontinuous drop in reliability (in the limit model). We can define a correspondence $\boldsymbol{\rho}(\boldsymbol{x})$ which is limit of the reliability functions[12] $\boldsymbol{\rho}(\boldsymbol{x}, \boldsymbol{\mu}(\tau))$ as $\tau \to \infty$. This is equal to $\widehat{\boldsymbol{\rho}}(\boldsymbol{x})$ except at the discontinuities, where $\widehat{\boldsymbol{\rho}}(\boldsymbol{x})$ is multi-valued. Paralleling the case of homogeneous firms, this implies the potential for endogeneous fragility. We show in Section SA5.3, via a numerical example, that this potential is realized: fragile outcomes do indeed obtain (for an open set of parameters).

[12]Recall (SA-8.

SA5.2. **Proof of Proposition 5 (Weakest link).** Here we prove Proposition 5 in the main text.

Part (i): Let $\mathcal{P}$ be a directed path of length T from node T to node i and denote product i by 1. Consider a shock that decreases all relationship strengths, $\boldsymbol{x}' < \boldsymbol{x}$ elementwise, and thus makes $r_1' = 0$ by definition of i being critical, where the "prime" notation denotes the quantity after the shock.

If any product $t+1$ sources input t and $r_t' = 0$, we have that

$$\begin{aligned} r_{t+1}' &= \prod_{l \in I(t+1)} (1 - (1 - x_{t+1,l}' r_l')^{n_{t+1,l}}) \\ &= 0 \end{aligned}$$

since $t \in I(t+1)$.

Since $r_1' = 0$, it then follows by induction that the production of all products $t \in \mathcal{P}$ will fail.

Part (ii): This follows immediately from Part (i) and the fact that i and j are in the same strongly connected component $\mathcal{I}^{SC}$ if and only if there is a directed path in the product dependence graph from each product to the other. □

SA5.2.1. *Endogenous investment.* So far we have discussed only the mechanics of the model. Now we explicitly model investment. For a firm if and an input $j \in I(i)$, the cost of effort is $c_{ij}(x_{if,j} - \underline{x}_{ij})$. The gross profit conditional on producing product i is $g_i(r_i)$. Here g_i is a product-specific function (which can capture many different features of different product markets that affect their profitabilities) and r_i is the reliability of producers of product i. We work here with a reduced-form specification of gross profits, and assume that all functions satisfy the same assumptions as their counterparts in the homogeneous model. As before, shocks are modeled as reducing $\underline{x}_{ij}$ after investments have been made.

Investment decisions are made simultaneously by firms, knowing their product type i but ex ante of depth realizations. Our solution concept is symmetric undominated equilibrium.

A full theoretical analysis of this model is beyond our scope, but in the following sections we show numerically that fragility of the type we have identified is consistent with equilibrium investment, and illustrate the other phenomena that arise in the model with heterogeneities.

SA5.3. **Examples with heterogeneities.**

SA5.3.1. *The mechanics of production.* There are seven products. Only product a is used as an input into its own production. Products a, b, c and d all use inputs from each other; products e, f, and g also all use inputs from each other, but also require product a as an input. Figure 7 shows the input dependencies between these products. There are two strongly connected components. Products a, b, c, and d form a strongly connected component while e, f, and g form another strongly connected component. Products a and b have three potential suppliers for each of their required inputs, while all other products have only two potential suppliers for each of their required inputs.

SA5.3.2. *Implications for criticality.* We can begin with some general observations that do not depend on any primitives beyond what has been described. By Proposition 5(ii), products within a strongly connected component must be all critical or all non-critical. By Proposition 5(i), if a given product is critical, all products that source this product directly or indirectly must also be critical. Combining these results, there are just three possibilities: no products are critical; only products e, f, g are critical; all products are critical. Note that it is impossible for products a, b, c, and d to be critical while products e, f, and g are non-critical. The reason is that if production of product a fails, then the e, f and g producers will be not be able to source all the inputs they need, and so will also be unable to produce. On the other hand, note that production of the a, b ,c and d products does not require sourcing any of the e, f, g products, so we can have the e, f, g products be critical while the a, b, c, d products are not.

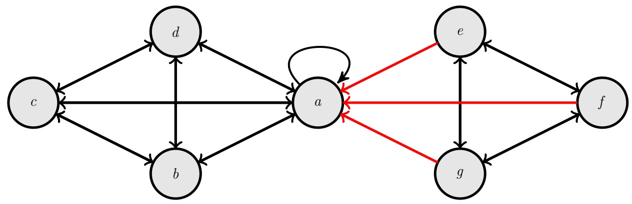


FIGURE 7. Supply dependencies: Bidirectional arrows represent reciprocated supply dependencies in which both products require inputs from each other. A red one-directional arrow from one product to another means that the product at the origin of the arrow uses as an input the product at the end of the arrow (e.g. product e requires product a as an input, but not the other way around). Product a also depends on itself, reflected by the loop. Note that a, b, c, d form a strongly connected component while e, f, g form another strongly connected component.

SA5.3.3. *An outcome at which a strict subset of firms is critical: Mechanics.* The system $\boldsymbol{r} = \mathcal{R}(\boldsymbol{r})$ can be written as, for all i

$$r_i = \prod_{j \in I(i)} (1 - (1 - x_{ij} r_j)^{n_{ij}}).$$

Here $I(i)$ is the neighborhood of i on the product dependency graph, with $|I(i)| = m_i$ (the complexity of production for product i), and n_{ij} is the number of potential suppliers a producer of product i has for input j (i.e. the potential level of multisourcing by producers of product i for input j).

We will now exhibit a point $(\boldsymbol{x}, \boldsymbol{r})$ at which products $\{a, b, c, d\}$ are critical in the limit model (and thus for large enough depths).[13] An entry x_{ij} in the following matrix represents the strength chosen by a producer of product i in a relationship sourcing input j.

$$\boldsymbol{x} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 0.8873 & 0.8872 & 0.9315 & 0.9385 & 0 & 0 & 0 \\ 0.8773 & 0 & 0.9204 & 0.9272 & 0 & 0 & 0 \\ 0.8673 & 0.8672 & 0 & 0.9084 & 0 & 0 & 0 \\ 0.8573 & 0.8572 & 0.8915 & 0 & 0 & 0 & 0 \\ 0.7573 & 0 & 0 & 0 & 0 & 0.9726 & 0.9783 \\ 0.7473 & 0 & 0 & 0 & 0.9464 & 0 & 0.9572 \\ 0.7373 & 0 & 0 & 0 & 0.9265 & 0.9317 & 0 \end{bmatrix} \end{array}. \quad \text{(SA-10)}$$

The product reliabilities $\boldsymbol{r}$ are as follows:

$$\boldsymbol{r} = [0.9926, 0.9928, 0.9387, 0.9307, 0.5384, 0.5262, 0.5145]. \quad \text{(SA-11)}$$

For this configuration, products e, f and g are critical, while a, b, c, and d are noncritical. To see this, we consider a small unanticipated shock to the relationship strengths $\boldsymbol{x}$ (e.g., we reduce all entries of $\boldsymbol{x}$ by a small δ). For firms producing products a, b, c or d, the impact of this is minor. The probability of successful production for those products (r_i) only drops continuously and for a small shock the change will be small. On the other hand, the small shock is sufficient for the output of the firms producing products e, f and g to collapse to 0. This can be seen by iteratively applying $\mathcal{R}$ after the small shock.

SA5.3.4. *An outcome at which a strict subset of firms is critical: Endogenous investment.* We now specify (heterogeneous) cost and profit functions that support the outcome described above as an equilibrium. Specifically, we let the gross profit function of a firm producing product i be

$$g_i(r_i) = \alpha_i(1 - r_i),$$

where

$$\alpha = [2857.4, 2456.1, 53.5, 43.9, 5.8, 5.5, 5.1].$$

We let the cost of a producer of product i from investing in supplier relationships with producers of product j be

$$c_{ij}(x_{ij} - \underline{x}_{ij}) = \frac{1}{2}\gamma_{ij} x_{ij}^2.$$

For simplicity, we have set $\underline{x}_{ij} = 0$ and we set $\gamma_{ij} = 1$ for all product pairs ij.

[13]At the end of SA5.3 we describe how this example was generated.

As we have mentioned, we consider symmetric equilibria, in the sense that all producers of product i invest the same amount sourcing a given input j. As we are interested in these symmetric investment choices, we dispense with the subscript f when we consider investments. In such an equilibrium the profit of a producer of product i is given by

$$\Pi_i = g_i(r_i)r_i - \frac{1}{2}\sum_{j \in I(i)} \gamma_{ij} x_{ij}^2, \tag{SA-12}$$

From these specifications, it can be checked numerically that the x_{ij} in (SA-10) maximize the profit function given the levels of reliability in (SA-11). Our discussion below will show how the parameters, by construction, satisfy the first-order conditions of agents' optimization problems.

SA5.3.5. *Finding a critical outcome.* We now describe how we reduced the search for an example to a simple numerical problem.

The marginal benefit a producer of product i receives from investing in its relationships with suppliers of input j is

$$MB_{ij} = g_i(r_i) \prod_{l \in I(i), l \neq j} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{ij} (1 - x_{ij} r_j)^{n_{ij}-1} r_j. \tag{SA-13}$$

The marginal cost for a producer of product i investing in a relationship with a supplier of input j is

$$MC_{ij} = x_{ij}. \tag{SA-14}$$

We look for $|\mathcal{I}| \times |\mathcal{I}|$ matrix $\boldsymbol{x}$ (i.e., the matrix containing the investment profiles for all products), with entries x_{ij} satisfying $MB_{ij} = MC_{ij}$, along with parameters $\boldsymbol{\alpha}$ supporting this outcome. We will view the $\boldsymbol{\alpha}$ as free parameters to be chosen in our search.

Fix a conjectured $\check{\boldsymbol{r}} \in \mathbb{R}^{|\mathcal{I}|}$ (we begin with the vector of ones) and arbitrary $x_{i1} \in (0,1)$ for all $i \in I$. The value of x_{i1} that equates the marginal benefits and marginal costs for a producer of product i's investment into sourcing product 1 is increasing in $g_i(r_i) = \alpha_i(1 - \check{r}_i)$, and covers the $[0,1]$ interval as α_i varies. We can thus choose the free parameter α_i to achieve the desired $x_{i1} \in (0,1)$. We claim that the choice of x_{i1} for all i and $\check{\boldsymbol{r}}$ then pins down the value of x_{ij} for all $j \neq 1$ in any best-response profile, as follows. We must have

$$\frac{MB_{ij}}{MB_{i1}} = \frac{MC_{ij}}{MC_{i1}}, \quad \text{for all } i, j$$

which can be expressed as

$$\frac{g_i(\check{r}_i) \prod_{l \in I(i), l \neq j} (1 - (1 - x_{il}\check{r}_l)^{n_{il}}) n_{ij} (1 - x_{ij}\check{r}_j)^{n_{ij}-1} \check{r}_j}{g_i(\check{r}_i) \prod_{l \in I(i), l \neq 1} (1 - (1 - x_{il}\check{r}_l)^{n_{il}}) n_{i1} (1 - x_{i1}\check{r}_1)^{n_{i1}-1} \check{r}_1} = \frac{x_{ij}}{x_{i1}},$$

and reduces to

$$\frac{(1 - x_{ij}\check{r}_j)^{n_{ij}-1}}{(1 - (1 - x_{ij}\check{r}_j)^{n_{ij}})} \frac{n_{ij}}{x_{ij}} = \frac{n_{i1}}{x_{i1}} \frac{\check{r}_1}{\check{r}_j} \frac{(1 - x_{i1}\check{r}_1)^{n_{i1}-1}}{(1 - (1 - x_{i1}\check{r}_1)^{n_{i1}})}. \tag{SA-15}$$

The left-hand side is decreasing in x_{ij} while the right-hand side is given, so there can be only one solution x_{ij} satisfying the above.

Setting expressions (SA-13) and (SA-14) equal to each other implies a value for $g_i(r_i)$ for all $i \neq 1$. Recall that $g_i(r_i) = \alpha_i(1 - r_i)$, and so depends on α_i. Thus, for a given value of $g_i(r_i)$ we set

$$\alpha_i = \frac{g_i(\check{r}_i)}{1 - \check{r}_i}.$$

This, along with our calculations above, ensures that all firms satisfy their first-order conditions for investment at reliabilties $\check{\boldsymbol{r}}$.

Now we describe the procedure that we use to search for an equilibrium where some products are critical.

- Initialize $(x_{i1})_{i \in \mathcal{I}}$ to values just smaller than 1 and decrease these values incrementally by a small amount along an arbitrary strictly decreasing curve.[14]
- For each $(x_{i1})_{i \in \mathcal{I}}$, repeat ...
  - calculate the entire $\boldsymbol{x}$ matrix using the above procedure;

[14] We chose the curve to find an example where one component of the network becomes critical before the whole network becomes critical.

    – update the conjectured reliabilities by updating $\check{\boldsymbol{r}} \leftarrow \widehat{\boldsymbol{\rho}}(\boldsymbol{x})$.

  ... until $\|\widehat{\boldsymbol{\rho}}(\boldsymbol{x}) - \check{\boldsymbol{r}}\|$ is within a desired tolerance of 0.

- Move on to the next $(x_{i1})_{i \in \mathcal{I}}$.

We continue until the probability of successful production decreases to 0 for one of the products i. We then look at the $(x_{i1})_{i \in \mathcal{I}}$ just before this has happened. This gives us a value of $\boldsymbol{x}$ (i.e., the investment profiles for all products) such that, for any given shock magnitude, at least one product has reliability 0 after the shock.

As we have discussed, the equations above give us parameters such that the first-order conditions for optimality hold. We can then check for global optimality numerically.

The above algorithm helped us find $(\boldsymbol{\alpha}_{\text{crit}}, \boldsymbol{x}_{\text{crit}}, \boldsymbol{r}_{\text{crit}})$ together constituting an equilibrium in the infinite-depth model, corresponding to $\widehat{\boldsymbol{\rho}}$ in Section SA5.1. In the finite-depth version of the model, the only difference is that $\widehat{\boldsymbol{\rho}}(\boldsymbol{x})$ should be replaced by $\boldsymbol{\rho}(\boldsymbol{x}, \boldsymbol{\mu})$ for deep[15] $\boldsymbol{\mu}$. On the part of $\widehat{\boldsymbol{\rho}}(\boldsymbol{x})$ visited by the algorithm, the finite-depth reliability and best-response functions converge to $\widehat{\boldsymbol{\rho}}(\boldsymbol{x})$ pointwise (and, by a compactness argument, therefore uniformly). It follows that the computations of the finite-depth algorithm converge uniformly to the ones defined here, under a standard nonsingularity condition at the equilibrium which can be checked numerically.[16]

SA5.3.6. *A different set of critical products.* As we have mentioned, there are essentially two types of equilibria with fragile firms. If a firm in the set $\{e, f, g\}$ becomes critical first, then all firms in this set simultaneously become critical, but not necessarily the others. This is the case we have seen above.

When a firm in the set $\{a, b, c, d\}$ becomes fragile first, a shock to any one of $\{a, b, c, d\}$ that reduces the reliability of sourcing an input is sufficient for the probability of successful production of *all* firms to fall to 0.

To illustrate the latter possibility, We adjust the configuration of the previous example by letting the vector of product profitabilities be

$$\alpha = [32.46, 45.37, 8.52, 9.24, 21.78, 24.62, 28.00].$$

Everything else remains the same as before.

The equilibrium investment levels are reported in the matrix $\boldsymbol{x}$ below.

$$\boldsymbol{x} = \begin{bmatrix} 0.7965 & 0.7792 & 0.8735 & 0.8663 & 0 & 0 & 0 \\ 0.8065 & 0 & 0.8859 & 0.8785 & 0 & 0 & 0 \\ 0.8165 & 0.8029 & 0 & 0.8681 & 0 & 0 & 0 \\ 0.8265 & 0.8124 & 0.8855 & 0 & 0 & 0 & 0 \\ 0.8965 & 0 & 0 & 0 & 0 & 0.8947 & 0.8894 \\ 0.9065 & 0 & 0 & 0 & 0.9103 & 0 & 0.8992 \\ 0.9165 & 0 & 0 & 0 & 0.9204 & 0.9146 & 0 \end{bmatrix}.$$

The reliabilities are as follows

$$\boldsymbol{r} = [0.8837, 0.9132, 0.7653, 0.7756, 0.8778, 0.8865, 0.8951].$$

This example is constructed analogously to the previous one, just tuning the starting point to reach a different critical outcome. Indeed, at these parameter values production of all products is now critical. Following a small shock to relationship strengths $\boldsymbol{x}$ (again, we reduce all relationship strengths a little) reliability of all products collapses to 0.

[15] We can each product's suppliers' depths to be drawn from a Poisson distribution with mean τ just for concreteness and let $\tau \to \infty$.

[16] Intuitively, paralleling the illustration of Theorem 1, the condition is that the equilibrium is not at a point where the two curves are tangent.

SA6. Interpretation of investment

SA6.1. **Effort on both the extensive and intensive margins.** This section supports the claims made in Section 6.2.1 of the paper that our model is easily extended to allow firms to make separate multi-sourcing effort choices on the intensive margin (quality of relationships) and the extensive margin (finding potential suppliers).

Suppose a firm if chooses efforts $\widehat{e}_{if} \geq 0$ on the extensive margin and effort $\widetilde{e}_{if} \geq 0$ on the intensive margin, and suppose that $x_{if} = h(\widehat{e}_{if}, \widetilde{e}_{if})$. Let the cost of investment be a function of $\widehat{e}_{if} + \widetilde{e}_{if}$ instead of y_{if}. This firm problem can be broken down into choosing an overall effort level $e_{if} = \widehat{e}_{if} + \widetilde{e}_{if}$ and then a share of this effort level allocated to the intensive margin, with the remaining share allocated to the extensive margin. Fixing an effort level e, a firm will choose $\widehat{e}_{if} \in [0, e]$, with $\widetilde{e}_{if} = e - \widehat{e}_{if}$, to maximize x_{if}. Let $\widehat{e}^*_{if}(e)$ and $\widetilde{e}^*_{if}(e) = e - \widehat{e}^*_{if}(e)$ denote the allocation of effort across the intensive and extensive margins that maximizes x_{if} given overall effort e. Given these choices, define $h^*(e) := h(\widehat{e}^*_{if}(e), \widetilde{e}^*_{if}(e))$. As h^* is strictly increasing in e, choosing e is then equivalent to choosing x_{if} directly, with a cost of effort equal to $c(h^{*-1}(e))$. Thus, as long as the cost function $\widehat{c}(e) := c(h^{*-1}(e))$ continues to satisfy our maintained assumptions on c, everything goes through unaffected.

SA6.2. **A richer extensive margin model.** In the previous subsection we gave an extensive margin search effort interpretation of x_{if}. In some ways this interpretation was restrictive. Specifically, it required there to be exactly n suppliers capable for supplying the input and that each such supplier be found independently with probability x_{if}. This alternative interpretation is a minimal departure from the intensive margin interpretation, which is why we gave it. However, it is also possible, through a change of variables, to see that our model encompasses a more general and standard search interpretation.

Fixing the environment a firms faces, specifically the probability other firms successfully produce $r > 0$ and a parameter n that will index the ease of search, suppose we let each firm if choose directly the probability that, through search, it finds an input of given type. When $r = 0$ we suppose that all search is futile and that firms necessarily choose $\widehat{x}_{if} = 0$. Denote the probability firm if finds a supplier of a given input type by $\widehat{x}_{if}$. Conditional on finding an input, we let it be successfully sourced with probability 1 so all frictions occur through the search process. Implicitly, obtaining a probability $\widehat{x}_{if}$ requires search effort, and we suppose that cost of achieving probability $\widehat{x}_{if}$ is $\widehat{c}(\widehat{x})$, where $\widehat{c}$ is a strictly increasing function with $\widehat{c}(0) = 0$.

We suppose firms choose $\widehat{x}_{if}$ taking the environment as given. In particular, firms take as given the probability that suppliers of the inputs they require successfully produce. When many potential suppliers of an input produce successfully we let it be relatively easy to find one, and if none of these suppliers produce successfully then it is impossible to find one. In addition, the parameter n shifts how easy it is to find a supplier.

Given this set up we can let the probability of finding a supplier have the functional form $\widehat{x} := 1 - (1 - x_{if} r)^n$, and the cost of achieving this probability be given by $\widehat{c}(\widehat{x}) := c\left(\frac{1-(1-\widehat{x})^{1/n}}{r}\right)$. Although these functional form assumptions might seem restrictive, we still have freedom to use any function c satisfying our maintained assumptions. This degree of freedom is enough for the model to be quite general as all that matters is the size of the benefits of search effort relative to its cost, and not the absolute magnitudes. Further, these functional form assumptions satisfy all the desiderata we set out above. As $1 - (1 - x_{if} r)^n$ is the key probability throughout our analysis, all our results then go through with this interpretation.

References

Baqaee, D. R. (2018): "Cascading Failures in Production Networks," *Econometrica*, 86, 1819–1838.

Echenique, F. (2005): "A short and constructive proof of Tarski's fixed-point theorem," *International Journal of Game Theory*, 33, 215–218.

Elliott, M., B. Golub, and M. O. Jackson (2014): "Financial Networks and Contagion," *American Economic Review*, 104, 3115–53.

Tintelnot, F., A. K. Kikkawa, M. Mogstad, and E. Dhyne (2018): "Trade and Domestic Production Networks," Working Paper 25120, National Bureau of Economic Research.