SUPPLEMENTARY APPENDIX
SUPPLY NETWORK FORMATION AND FRAGILITY

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This document contains supporting material for the document "Supply Network Formation and Fragility," which herein we refer to as the "main document."

1. Omitted Proofs

1.1. **Lemma 2.** Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \tag{1}$$

Then there are values $x_{\text{crit}}, r_{\text{crit}} \in (0, 1]$ such that:

(i) $\rho(x) = 0$ for all $x < x_{\text{crit}}$;
(ii) ρ has a (unique) point of discontinuity at x_{crit};
(iii) ρ is strictly increasing for $x \geq x_{\text{crit}}$;
(iv) the inverse of ρ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[r_{\text{crit}}, 1]$, where $r_{\text{crit}} = \rho(x_{\text{crit}})$;
(v) χ is positive and quasiconvex on the domain $(0, 1]$;
(vi) $\chi'(r_{\text{crit}}) = 0$.

Proof. We first list some properties of ρ and χ.

Property 0: For positive r in the range of ρ, we have $r = \rho(x)$ if and only if

$$x = \frac{1 - (1 - r^{1/m})^{1/n}}{r}.$$

This is shown by rearranging equation PC in the main document.

Property 1: $\chi(1) = 1$. This follows by inspection.

Property 2: $\chi(r) > 0$ for all $r \in (0, 1]$. This follows by inspection.

Property 3: $\lim_{r \downarrow 0} \chi(r) = \infty$. This follows by an application of l'Hopital's rule, i.e.

$$\lim_{r \downarrow 0} \frac{\frac{d}{dr}\left(1 - (1 - r^{1/m})^{1/n}\right)}{\frac{d}{dr} r} = \lim_{r \downarrow 0} \frac{(1 - r^{1/m})^{1/n - 1} r^{1/m - 1}}{mn} = \infty.$$

Property 4: $\lim_{r \uparrow 1} \chi'(r) = \infty$. This follows by examining

$$\chi'(r) = \frac{r^{1/m}(1 - r^{1/m})^{1/n-1}}{mnr^2} + \frac{(1 - r^{1/m})^{1/n} - 1}{r^2}$$

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Property 5: There is a unique interior $r_{\text{crit}} \in (0,1)$ minimizing $\chi(r)$. To show this, define $z(r) = 1/(1-r^{1/m})$, and note that $r \in (0,1)$ satisfies $\chi'(r) = 0$ if and only if the corresponding $z(r) > 1$ solves

$$z - 1 = mn(z^{1/n} - 1);$$

here we use that the function z is a bijection from $(0,1)$ to $(1,\infty)$. The equation clearly has exactly one solution for $z > 1$.[1] Now it remains to see that the unique $r \in (0,1)$ solving $\chi'(r) = 0$ defines a local minimum. Note that χ is continuously differentiable on $(0,1)$. Property 3 implies that $\chi'(r) < 0$ for some $r < r_{\text{crit}}$ while Property 4 implies that $\chi'(r) > 0$ for some $r > r_{\text{crit}}$. These points suffice to show Property 5.

To prove the lemma, we relate desired properties of ρ to properties of χ; the claims made here can be visualized by referring to Figure 10 in the main document, panels (A) and (C), which illustrate the properties of the functions involved.

Together, Properties 0 and 5 imply that there is no $r > 0$ in the range of ρ such that $x < \chi(r_{\text{crit}})$. Let $x_{\text{crit}} = \chi(r_{\text{crit}})$, which yields (vi) of the lemma; then what we have said implies $\rho(x) = 0$ for $x < x_{\text{crit}}$, i.e., statement (i) of the lemma. The proof of Property 5 also implies statement (v) in the Lemma.

It remains to show (ii-iv) of the Lemma. By definition, $\rho(x)$ is the largest solution of PC. Properties 1 and 5 imply that on the domain $[r_{\text{crit}}, 1]$, χ is a strictly increasing function whose range is $[x_{\text{crit}}, 1]$. Fix an $r \in [r_{\text{crit}}, 1]$ and let $x = \chi(r)$. What we have said implies that (x, r) solves PC and that there is no r' with $r' > r$ such that (x, r') solves PC. Thus, by definition $r = \rho(x)$. Notice that as we vary r in the interval, x varies over the interval $[x_{\text{crit}}, 1]$. This establishes (iv), and (iii) follows immediately from the fact that χ is increasing on the domain in question. For (ii), it suffices to deduce from Properties 2 and 5 that the minimum of χ has both coordinates positive.

We now consider the case $n = 1$. Here, $\frac{d\chi(r)}{dr} = \frac{r^{1/m} - mr^{1/m}}{mr^2} < 0$ for all $r \in (0,1]$. Since from Properties 1 and 3 (which still hold when $n = 1$), $\chi(1) = 1$ and $\lim_{r \downarrow 0} \chi(r) = \infty$, it follows that $\chi(r)$ is decreasing in r and has image $[1, \infty)$. Thus, by logic similar to the above, $\rho(x) = 0$ for all $x \in [0,1)$ and $\rho(x) = 1$ when $x = 1$. It follows that $x_{\text{crit}} = 1$ in this case and all the statements of the lemma are satisfied, though some of them are trivial. □

1.2. **Lemma 3.** Fix any $m \geq 2$, $n \geq 2$, and $x \geq x_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n, and x) such $0 \leq x_1 < x_2 < 1/\rho(x)$ so that:

0. $Q(0; x) = Q(1/\rho(x); x) = 0$ and $Q(x_{if}; x) > 0$ for all $x_{if} \in (0, 1/\rho(x))$;
1. $Q(x_{if}; x)$ is increasing and convex in x_{if} on an interval $[0, x_1]$;
2. $Q(x_{if}; x)$ is increasing and concave in x_{if} on an interval $(x_1, x_2]$;
3. $Q(x_{if}; x)$ is decreasing in x_{if} on an interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

[1] The left-hand side is linear and the right-hand side is concave, since $n \geq 2$. At $z = 1$ the two sides are equal, and the lines defined by the left-hand and right-hand sides are not tangent, so there is exactly one solution $z > 1$.

Proof. As a piece of notation, define

$$\zeta(x_{if}; x) = 1 - x_{if}\rho(x).$$

When using ζ, we will often omit the arguments for brevity. Then

$$Q(x_{if}; x) = mn\rho(x)\zeta^{n-1}(1-\zeta^n)^{m-1},$$

from which it follows immediately that $Q(0; x) = Q(1/\rho(x); x) = 0$ and $Q(x_{if}; x) > 0$ for all $x_{if} \in (0, 1/\rho(x))$. This establishes Property 0 in the lemma statement.

Next, we can calculate

$$Q'(x_{if}; x) = -mn\rho(x)^2\zeta^{n-2}(1-\zeta^n)^{m-2}[(mn-1)\zeta^n - n + 1].$$

Note that for $x_{if} \in (0, 1/\rho(x))$ we have[2]

$$\text{sign}[Q'(x_{if}, x)] = \text{sign}[(mn-1)\zeta^n - n + 1]. \tag{2}$$

Further, for sufficiently small $x_{if} > 0$

$$\text{sign}[(mn-1)\zeta^n - n + 1] = \text{sign}[n(m-1)] > 0.$$

Thus $Q'(0; x) > 0$.

We will now deduce from the above calculations about Q' that there is exactly one local maximum of $x_{if} \mapsto Q(x_{if}; x)$ on its domain, $[0, 1/\rho(x)]$. First, as this is a continuous function with $Q(0; x) = 0 = Q(1/\rho(x); x)$ and $Q'(0; x) > 0$, it follows there is an interior maximum of $Q(x_{if}; x)$ in the interval $(0, 1/\rho(x))$. Next, by (2), $\text{sign}[Q'(x_{if}, x)] > 0$ if and only if

$$x_{if} < \frac{1 - \left(\frac{n-1}{mn-1}\right)^{\frac{1}{n}}}{\rho(x)}.$$

Thus, there can be at most one value of x^*_{if} with $Q'(x^*_{if}; x) = 0$. Together, these observations imply that Q has one local optimum on its extended domain, which is in fact a global maximum. We let x_2 be defined by the unique value of x_{if} at which $Q'(x^*_{if}; x) = 0$. This establishes Property 3. It also establishes the "increasing" part of Properties 1 and 2, since $Q(x_{if}; x)$ is increasing to the left of x_2 by what we have said.

The next part of the proof studies the second derivative of Q to establish the claims about the convexity/concavity of Q. First note that

$$Q''(x_{if}; x) = mn\rho(x)^3\zeta^{n-3}(1-\zeta^n)^{m-3}H$$

where

$$H = \underbrace{\left(m^2n^2 - 3mn + 2\right)}_{A}\zeta^{2n} + \underbrace{\left((1-3m)n^2 + (3m+3)n - 4\right)}_{B}\zeta^n + \underbrace{n^2 - 3n + 2}_{C}.$$

For $x_{if} \in (0, 1/\rho(x))$, we can see that

$$\text{sign}[Q''(x_{if}; x)] = \text{sign}[H]. \tag{3}$$

Let $z := \zeta^n$ for $z \in (0,1)$ (which corresponds to $x_{if} \in (0, 1/\rho(x))$). We can then write $H = \widetilde{H}(z)$ for $z \in (0,1)$, where

$$\widetilde{H}(z) = Az^2 + Bz + C, \tag{4}$$

[2]The sign operator is +1 for positive numbers, −1 for negative numbers, and 0 when the argument is 0.

and A, B and C are constants (labeled above) depending only on m, n. $\widetilde{H}$ is therefore a quadratic polynomial in z and its roots depend only on n and m. Further, $A > 0$, $B < 0$, $C > 0$ and $A + B + C > 0$. Thus $\widetilde{H}$ is convex in z with $\widetilde{H}(0) > 0$ and $\widetilde{H}(1) > 0$.

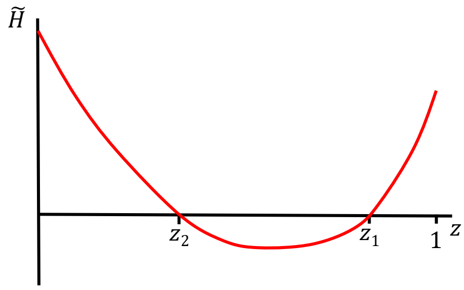
We first argue that $\min_{z\in[0,1]} \widetilde{H}(z) < 0$. Towards a contradiction suppose $\min_{z\in[0,1]} \widetilde{H}(z) \geq 0$. This implies that Q'' is nonnegative by equation 3 and hence that Q is globally convex. However, we have already established that $Q'(0; x) > 0$, so the convexity of Q implies there can be no interior maximum which contradicts our deductions above.

An immediate implication of $\min_{z\in[0,1]} \widetilde{H}(z) < 0$ is that $\widetilde{H}(z)$ has two real roots, z_1 and $z_2 < z_1$. This establishes the basic shape of $\widetilde{H}(z)$ as illustrated in Figure 1.

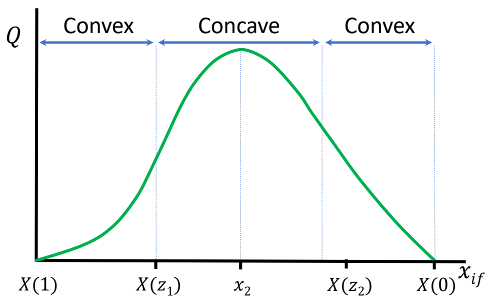
It will be helpful to sometimes consider the values of x_{if} that correspond to the roots of the $\widetilde{H}(z)$. To this end, we define the function

$$X(z) := \frac{1 - z^{1/n}}{\rho(x)}. \tag{5}$$

We can then set $x_1 = X(z_1)$ (i.e., the first inflection point of Q). Along with what we already know, the deduced shape of $\widetilde{H}(z)$ pins down the remaining properties we require about the shape of Q, as we now argue. For an illustration, see Figure 1.



(A)



(B)

FIGURE 1. Panel (a) shows basic shape of the function $\widetilde{H}(z)$. Panel(b) shows the basic shape of the function $Q(x_{if}; x)$, where the convexity and concavity in different regions is implied by the shape of $\widetilde{H}(z)$.

As $z_1 > z_2$, $\widetilde{H}'(z_1) > 0$. This corresponds to $Q''(x_{if}; x)$ going from positive to negative as x_{if} crosses $x_1 := X(z_1)$, and thus $Q(x_{if}; x)$ going from convex to concave. As $Q'(0; x) > 0$ and $Q(x_{if}; x)$ is convex for $x_{if} \in [0, x_1]$, the maximum of Q must occur at a value of $x_{if} \in (x_1, 1)$. Recalling that we let x_2 be the value of x_{if} at which $Q(x_{if}; x)$ is maximized we conclude that $x_1 < x_2$. Further, as for values of $x_{if} < x_1$ the function $Q(x_{if}; x)$ is convex we have established the convexity part of Property 2 of Lemma 3.

By similar reasoning, $\widetilde{H}'(z_2) < 0$. This corresponds to $Q''(x_{if}; x)$ going from negative to positive as x_{if} crosses $X(z_2)$. Recall that x_2 is defined as the maximum of Q. We must have $X(z_2) > x_2$. If not, Q would be increasing and convex for all $x_{if} \geq X(z_2)$, contradicting the existence of an interior maximum as already established. This establishes that the function Q remains concave until after its maximum point x_2 establishing the concavity part of Property 3.

We now argue that $x_1 < x_{\text{crit}}$ to establish Property 4. We do so in two steps. First we show it for the case in which firms other than if are investing at the critical level, i.e., $x = x_{\text{crit}}$, and then for $x > x_{\text{crit}}$.

First then, we need to establish that

$$x_1 = \frac{1 - z_1^{1/n}}{\rho(x_{\text{crit}})} < \frac{1 - (1 - \rho(x_{\text{crit}})^{1/m})^{1/n}}{\rho(x_{\text{crit}})} = x_{\text{crit}}.$$

Writing $\rho(x_{\text{crit}})$ as r_{crit}, this holds if and only if $z > 1 - r_{\text{crit}}^{1/m}$, which is equivalent to $r_{\text{crit}} > (1-z)^m$. A sufficient condition for this to hold is that $\chi'((1-z)^m) < 0$, since we know from Lemma 2 in the main document that $\chi'(r_{\text{crit}}) = 0$.

Note that

$$\chi'(r) = \frac{\frac{r^{1/m}(1-r^{1/m})^{1/n-1}}{mn} + (1 - r^{1/m})^{1/n} - 1}{r^2}$$

Putting $r = (1-z)^m$ in the above yields

$$\chi'((1-z)^m) = \frac{\frac{(1-z)z^{1/n-1}}{mn} + (z)^{1/n} - 1}{(1-z)^{2m}}.$$

The denominator is always positive, hence we only need to verify that the numerator is negative, when evaluated at the larger of the two roots of $\widetilde{H}$, which we will call z_1. The numerator simplifies to

$$Num(z) = z^{1/n}\left(\frac{1-z}{mnz} + 1\right) - 1$$

and must now be evaluated at the root z_1. The latter is given by the quadratic formula as

$$z_1 = \frac{D + \sqrt{D^2 - 4(n^2 - 3n + 2)(m^2n^2 - 3mn + 2)}}{2(m^2n^2 - 3mn + 2)},$$

where $D = 3mn^2 - 3mn - n^2 - 3n + 4$. This expression for z_1 simplifies to

$$z_1 = \frac{n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn - m - n - 7)} - n - 3\Big) + 4}{2(mn-2)(mn-1)}.$$

Now note that the shape of $Num(z)$ is pictured in Fig. 2:

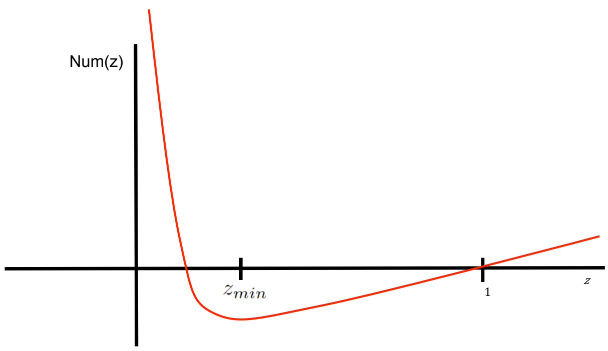


FIGURE 2. Shape of $Num(z)$.

Thus a sufficient condition for $Num(z_1) < 0$ is that $z_1 > z_{min}$.

It is easy to find $z_{min} = \frac{n-1}{mn-1}$ by solving $\frac{dNum(z)}{dz} = 0$. Indeed,

$$\frac{dNum(z)}{dz} = \frac{z^{1/n-2}(z(mn-1)+1-n)}{mn^2}$$

and thus a first-order condition is satisfied when $z(mn-1)+1-n=0$.

We also note that since $z^{1/n-2}$ and mn^2 are always positive,

$$\text{sign}\left[\frac{dNum(z)}{dz}\right] = \text{sign}\left[z(mn-1)+1-n\right],$$

and thus the function $Num(z)$ is decreasing on $z \in [0, z_{min})$ and increasing on $z \in (z_{min}, \infty)$.

Since $Num(1) = 0$, the above imply that $Num(z) < 0$ over the range $[z_{min}, 1)$. Thus, we only need to check that $z_1 > z_{min}$ in order to guarantee that $Num(z_1) < 0$.

It is easy to verify that the following inequality holds:

$$\begin{aligned} z_{min} &= \frac{n-1}{mn-1} \\ &< \frac{n\Big(3m(n-1)+\sqrt{(m-1)(n-1)(5mn-m-n-7)}-n-3\Big)+4}{2(mn-2)(mn-1)} = z_1. \end{aligned}$$

Indeed, multiplying both sides by $2(mn-2)(mn-1)$ yields

$$2(mn-2)(n-1) < n\Big(3m(n-1)+\sqrt{(m-1)(n-1)(5mn-m-n-7)}-n-3\Big)+4$$

which, after some algebra, reduces to

$$0 < n(mn+1-m-n) + n\sqrt{(m-1)(n-1)(5mn-m-n-7)}.$$

Since all terms on the right-hand side are clearly positive when $m, n \geq 2$, the inequality holds and we conclude that $z_1 > 1 - r_{\text{crit}}^{1/m}$. This establishes that $\widehat{x} < x_{\text{crit}}$ when $x = x_{\text{crit}}$.

We now consider $x > x_{\text{crit}}$. From Eq. (5), it is clear that x_1 is decreasing in $\rho(x)$. Since from Lemma 2 in the main document $\rho(x)$ is increasing in x, for $x \geq x_{\text{crit}}$, it the follows that x_1 is decreasing in x. This establishes Property 4 of Lemma 3, completing the proof of Lemma 3.

□

1.3. **Lemma 5.** The function $\beta(r)$ is quasiconcave and has a maximum at $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Proof. Recall that $\beta(r) = mnr^{2-\frac{1}{m}}(1-r^{\frac{1}{m}})^{1-\frac{1}{n}}$. We will show that $\beta(r)$ is quasiconcave by demonstrating that there exists an $\hat{r} \in (0,1)$ such that $\beta'(r) > 0$ for $r \in (0, \hat{r})$, $\beta'(r) < 0$ for $r \in (\hat{r}, 1)$ and $\beta'(r) = 0$ for $r = \hat{r}$.

$$\beta'(r) = mn\left(2-\frac{1}{m}\right)r^{1-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} - \left(1-\frac{1}{n}\right)mnr^{2-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{-\frac{1}{n}}\left(\frac{1}{m}r^{\frac{1}{m}-1}\right)$$

$$= \quad mnr^{1-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{-\frac{1}{n}}\left[\left(2-\frac{1}{m}\right)-r^{\frac{1}{m}}\left[\left(2-\frac{1}{m}\right)+\left(1-\frac{1}{n}\right)\frac{1}{m}\right]\right].$$

Note that the the first factor, $mnr^{1-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{-\frac{1}{n}}$, exceeds 0 for any $r \in (0,1)$ and is equal to 0 for $r = 0, 1$. Moreover, the expression multiplying it,

$$\left(2-\frac{1}{m}\right)-r^{\frac{1}{m}}\left[\left(2-\frac{1}{m}\right)+\left(1-\frac{1}{n}\right)\left(\frac{1}{m}\right)\right],$$

is a strictly decreasing, continuous function of r. It is also strictly positive when $r = 0$ and strictly negative when $r = 1$, implying that it is equal to 0 at some $r \in (0,1)$. It follows that there exists an $\hat{r} \in (0,1)$ satisfying the claimed properties.

Having established that $\beta(r)$ is quasiconcave and $\beta'(\hat{r}) = 0$, it follows immediately that $\beta(r)$ is maximized at $r = \hat{r}$. To find $\hat{r}$, we use its defining property and solve the following equation:

$$r\left((2m-1)nr^{-1/m}-2mn+1\right)\left(1-r^{1/m}\right)^{-1/n}=0.$$

As $r\left(1-r^{1/m}\right)^{-1/n} > 0$ for any positive production equilibrium, the equation is solved by

$$\hat{r}=\left(\frac{(2m-1)n}{2mn-1}\right)^{m}.$$

□

1.4. **Lemma 6.** Recall that $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^{m}$. For all $n \geq 2$ and $m \geq 3$, $\hat{r} < r_{\text{crit}}$.

Proof. By Lemma 2 in the main document, the function $\chi(r)$ is positive and quasiconvex on the domain $[0,1]$ with $r_{\text{crit}} = \operatorname{argmin}_r \chi(r)$. Thus, $\hat{r} < r_{\text{crit}}$ if and only if $\chi'(\hat{r}) < 0$.

Recall that $\chi(r) = \frac{1-(1-r^{1/m})^{1/n}}{r}$ and so

$$\chi'(r)=\frac{\frac{1}{n}(1-r^{1/m})^{1/n-1}\frac{1}{m}r^{1/m}-\left(1-(1-r^{1/m})^{1/n}\right)}{r^{2}}.$$

We will study this expression evaluated at $\hat{r} = \left(\frac{(2m-1)n}{2mn-1}\right)^{m}$ from Lemma 5 in the main document.

The denominator of the above expression for $\chi'(r)$ is always positive, so we need only check that the numerator is negative. Calling $A = \frac{n-1}{2mn-1}$ and $B = \frac{n+1-1/m}{n-1}$, we may rewrite the numerator, after some simplifications, as $A^{1/n}B - 1$, and thus we need only check that

$$A^{1/n}B < 1 \tag{6}$$

Let $h(n,m) = A^{1/n}B$ .To demonstrate (6), we will show that:

- Step 1: $h(n,3) < 1$, for all $n \geq 2$.
- Step 2: $h(n,m)$ is decreasing in m, for all $n \geq 2$.

Step 1: First note that

$$h(n,3)=\left(\frac{n-1}{6n-1}\right)^{1/n}\left(\frac{n+1-1/3}{n-1}\right)$$

We will show that

- Step 1a: $h(n,3)$ is increasing in n.
- Step 1b: $\lim_{n\to\infty} h(n,3) = 1$.

From this we can conclude that $h(n,3) < 1$, for all $n \geq 2$.

To show Step 1a, note that

$$\frac{\partial h(n,3)}{\partial n} = -\frac{\left(\frac{n-1}{6n-1}\right)^{1/n}\left((18n^2+9n-2)\ln(\frac{n-1}{6n-1}) + 30n^2 + 10n\right)}{3n^2\frac{(n-1)}{(6n-1)}}$$

The denominator is positive for all $n \geq 2$. Looking at the numerator, since $\left(\frac{n-1}{6n-1}\right)^{1/n} > 0$ for all $n \geq 2$, it suffices to show that

$$(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n < 0 \quad \text{for all } n \geq 2$$

in order to ensure that $\frac{\partial h(n,3)}{\partial n} > 0$. It is easy to show that

$$\ln\left(\frac{n-1}{6n-1}\right) < \ln\left(\frac{1}{6}\right) < -1.79$$

Thus, for all $n \geq 2$

$$\begin{aligned}(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n &< -(18n^2+9n-2)1.79 + 30n^2 + 10n \\ &< -32n^2 - 16n + 4 + 30n^2 + 10n \\ &= -2n^2 - 6n + 4 \\ &< 0.\end{aligned}$$

We thus conclude[3] that $h(n,3)$ is increasing in n and Step 1a is proved.

Step 1b follows immediately by noting that

$$\lim_{n\to\infty} h(n,3) = \lim_{n\to\infty}\left(\frac{1}{6}\right)^{1/n} = 1$$

We have thus proved Step 1.

Step 2. To show that $h(n,m)$ is decreasing in m, let us note that, for all $n \geq 2$

$$\begin{aligned}\frac{\partial h(n,m)}{\partial m} &= B\frac{\partial A^{1/n}}{\partial m} + A^{1/n}\frac{\partial B}{\partial m} \\ &= \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-2}{m(n-1)}\frac{mn(1+1/n)-1}{2mn-1} + \frac{1}{(n-1)m^2}\right) \\ &< \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-1}{m(n-1)} + \frac{1}{(n-1)m^2}\right) \\ &< 0.\end{aligned}$$

where the second equality follows after some simplifications, while the first inequality follows from the fact that $\frac{mn(1+1/n)-1}{2mn-1} > \frac{1}{2}$, which is easy to check.

We have thus shown that $h(n,m)$ is decreasing in m, for any $n \geq 2$, and Step 2 is thus proved.

[3]It is worth noting that this argument does not work for $m = 2$, in which case the numerator of $\frac{\partial h(n,2)}{\partial n}$ could be negative and thus $h(n,2)$ could be decreasing.

This concludes the proof of the lemma. □

1.5. **Lemma 7.** The function $x^*(\bar{f})$ is decreasing in $\bar{f}$ on any interval of values of $\bar{f}$ where $x^*(\bar{f})$ is positive. Moreover, $G(r\bar{f})$ is also decreasing in $\bar{f}$ on an interval of values of $\bar{f}$ where r is positive.

Proof. In this proof, we take r, x, g, and a new function we define called h all to be implicit functions of $\bar{f}$ at the unique positive investment equilibrium.

Consider the following system of equations in the variables r, x, g, h, and x:

$$x = \chi(r) \tag{7}$$

$$g = g(r\bar{f}) \tag{8}$$

$$h = r^{2-\frac{1}{m}}\left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} \tag{9}$$

$$kgh = c'(x) \tag{10}$$

where k is the constant $k = \kappa mn$. The first equation is, by Lemma 2 from the main document, equivalent to physical consistency positive investment equilibrium. The second is the definition of g.[4] The third is a definition of an auxiliary symbol h that allows us to write the final equation parsimoniously, which is the condition that marginal benefits are equal to marginal cost at an equilibrium.

At a positive investment equilibrium, this system of equations is satisfied by some values of the variables. Considering an interval of values of $\bar{f}$ where a positive investment equilibrium exists, we will implicitly differentiate this system to study the comparative statics of the equilibrium, which is valid because the equations are necessary conditions at the unique positive investment equilibrium. Write a dot over an endogenous variable to denote its derivative in $\bar{f}$. We make several observations following from the equations displayed above.

1. $\dot{r}$ and $\dot{x}$ have the same sign,[5] by equation (7) and Lemma 2 in the main document.
2. $\dot{h}$ has a sign opposite[6] of $\dot{r}$ (and $\dot{x}$) because by equation (9) and Lemma 5 in the main document, $\dot{h}$ is a decreasing function of h for a positive investment equilibrium.
3. $\dot{g} = g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]$, using the chain rule on $g = g(r\bar{f})$.
4. Differentiating $kgh = c'(x)$ implicitly and using point 3, we have

$$k(g\dot{h} + \dot{g}h) = c''(x)\dot{x}$$
$$k(g\dot{h} + \left[g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]\right]) = c''(x)\dot{x}.$$

Now, suppose toward a contradiction that $\dot{x} \geq 0$. The right-hand side is then weakly positive because c'' is positive. The $g\dot{h}$ term is weakly negative by point 2. The $\left[g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]\right]$ is strictly negative because $r > 0$ and $\dot{r} \geq 0$. This is a contradiction.

Now to show that $G(r\bar{f})$ is also decreasing in $\bar{f}$, for all values of $\bar{f}$ where r is positive, simply note from point 4 that

$$k\dot{g}h = c''(x)\dot{x} - kg\dot{h}$$

[4] We have abused notation by using g for the function value as well as the function.

[5] That is, one is positive if and only if the other is positive, and if one is zero then the other is.

[6] If one is positive, the other is negative. If one is zero, so is the other.

where $c''(x)\dot{x} < 0$ and $kg\dot{h} > 0$ (from point 2). Thus the right-hand side is negative. This then implies that $\dot{g} < 0$, since k and h are positive. The result thus immediately follows.

□

1.6. **Lemma 8.** *H* has the following properties:

- $H(0) > 0$;
- $H(1) < 1$;
- $H(\bar{f})$ is strictly decreasing for all $\bar{f}$ such that $x^*(\bar{f}) > 0$, and $H(\bar{f}) = 0$ for all $\bar{f}$ such that $x^*(\bar{f}) = 0$.

Proof. Recall the definition

$$H(\bar{f}) = \Phi^{-1}\big(\max\{G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})), 0\}\big).$$

Suppose the entry cutoff $\bar{f}$ is in effect and $x^*(\bar{f}) \geq x_{\text{crit}}$ is the positive level of investment. We show that the profits any firm if makes by entering the market are decreasing in $\bar{f}$. To this end, consider two values $\bar{f}', \bar{f}''$ such that $\bar{f}'' < \bar{f}'$, $x^*(\bar{f}) \geq x_{\text{crit}}$ when $\bar{f}$ is either of the two. For profits upon entering and investing x_{if} to be non-negative for firm if given an entry level $\bar{f}$, it must be the case that

$$G(\bar{f}\rho(x^*(\bar{f})))(1-(1-x_{if}\rho(x^*(\bar{f})))^n)^m - c(x_{if}) - \Phi(f) \geq 0. \tag{11}$$

By Lemma 7 in the main document, when $x^*(\bar{f})$ is positive, both $\rho(x^*(\bar{f}))$ and $G(\bar{f}\rho(x^*(\bar{f})))$ are decreasing in $\bar{f}$. Thus, after a reduction in entry from $\bar{f}'$ to $\bar{f}''$, firm if can always deviate from the new equilibrium investment level by continuing to choose $x_{if} = x^*(\bar{f}')$—the optimal investment level prior to the reduction. With this choice, the positive profit constraint given by inequality (11) becomes slacker. Re-optimizing its investment choice, firm if's profits from entering can only weakly increase causing inequality (11) to become weakly slacker still. Thus equilibrium profits, before entry costs, are strictly decreasing in $\bar{f}$ when they are non-negative. As Φ is strictly increasing in its argument, so is Φ^{-1}. Thus when $x^*(\bar{f}) > 0$, $H(\bar{f})$ is strictly decreasing in $\bar{f}$.

As there exists a positive and symmetric equilibrium by assumption, and equilibrium profits are decreasing in $\bar{f}$ when they are non-negative, we must have $G(0)\rho(x^*(0)) - c(x^*(0)) > 0$. As $\Phi(0) = 0$ and Φ is a strictly increasing function, this proves that $H(0) > 0$. By Assumption 4 in the main document, $H(1) < 1$.

We have shown that $H(\bar{f})$ is strictly decreasing in $\bar{f}$ when $x^*(\bar{f}) > 0$. We show now that if $x^*(\bar{f}) = 0$ then $H(\bar{f}) = 0$. This follows immediately from the definition of $H(\bar{f})$ because $\rho(0) = 0$. □

References