

SUPPLEMENTARY APPENDIX SUPPLY NETWORK FORMATION AND FRAGILITY

MATTHEW ELLIOTT, BENJAMIN GOLUB, AND MATTHEW V. LEDUC

This document contains supporting material for the paper “Supply Network Formation and Fragility,” which herein we refer to as the “main paper” or simply “paper.”

SA1. FINITE PRODUCTION TREES

We continue here the discussion of Section 4.5.2. We focus on the case where there are many stages of production. The main results of the paper are largely concerned with the sharp discontinuity in $\rho(x)$ at x_{crit} when $m \geq 2$ and the fact that outcomes can endogenously approach the point of discontinuity. In this section we show that when we consider models with a finite number of tiers, the key properties of the model extend and, in particular, sufficiently (but finitely) complex economies are fragile to arbitrarily small shocks in equilibrium.

To motivate the literal modeling of assembly in finitely many tiers, we return to the example of an Airbus A380. This product has 4 million parts. The final assembly in Toulouse, France, consists of six large components coming from five different factories across Europe: three fuselage sections, two wings, and the horizontal tailplane. Each of these factories gets parts from about 1500 companies located in 30 countries¹. Each of those companies itself has multiple suppliers, as well as contracts to supply and maintain specialized factory equipment, etc.

First, we numerically investigate what the reliability function of the economy looks like if we truncate the production tree at a certain depth, as well as allowing for some systematic heterogeneity (for example, more upstream tiers being simpler). Then we consider a version of the finite-tier model with endogenous investment. There, we can extend our approach, solving for equilibrium in terms of a simple fixed point system.

SA1.1. The shape of reliability function, numerically. We consider a T -tier tree where each firm in tier t requires m_t kinds of inputs and has n_t potential suppliers of each input. The nodes at the final tier are functional for sure. We denote by $\rho_T(x)$ the probability of successful production at the top node of a T -tier tree with these properties. This is defined as

$$\rho_T(x) = (1 - (1 - \rho_{T-1}(x)x)^{n_T})^{m_T}$$

with $\rho_1(x) = 1$, since the bottom-tier nodes do not need to obtain inputs.

We see that the expression is recursive and, if unraveled explicitly, would be unwieldy after a number of tiers. However, we will see in the next subsection that if $m_t = m$ and $n_t = n$ for all t , then for any $x \in [0, 1]$, as T goes to infinity, $\rho_T(x)$ converges to $\rho(x)$, which is defined as the largest fixed point of equation (PC).

Date Printed. March 11, 2020.

¹Source: “FOCUS: The extraordinary A380 supply chain”. *Logistics Middle East*. Retrieved on 28 may, 2019 from <https://www.logisticsmiddleeast.com/article-13803-focus-the-extraordinary-a380-supply-chain>

We start with some examples where m_t and n_t are the same throughout the tree. Figure 1 illustrates the successful production probability $\rho_T(x)$ for different finite numbers of tier T and how quickly it converges to the discontinuous curve $\rho(x)$.

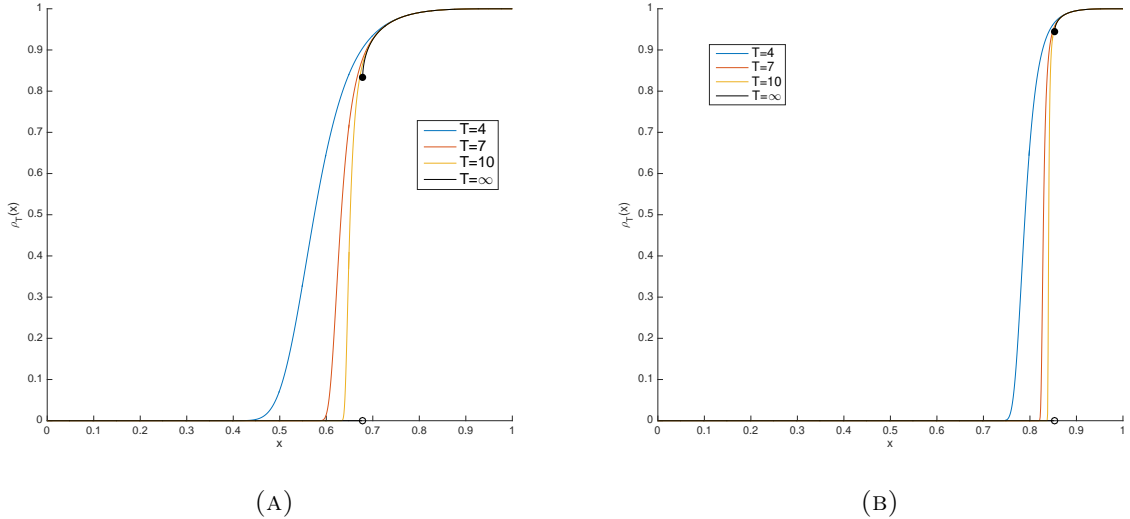


FIGURE 1. Successful production probability $\rho_T(x)$ for different finite numbers of tiers T . In panel (A), $m = 5$ and $n = 4$. In panel (B), $m = 40$ and $n = 4$.

In panel (A), we see that $\rho_T(x)$ exhibits a sharp transition for as few as 4 tiers. The red curve ($T = 7$) shows that when the investment level x drops from 0.66 to 0.61, or about 7 percent, production probability $\rho_7(x)$ drops from 0.8 to 0.1. (Thus ρ_7 achieves a slope of at least 14.) In panel (B), we see that increasing product complexity (by increasing m to 40) causes ρ_T to lie quite close to ρ . This illustrates how complementarities between inputs play a key role in driving this sharp transition in the probability of successful production. Note that $m = 40$ is not an exaggerated number in reality. In the Airbus example described earlier, many components would exhibit such a level of complexity.

However, a complexity number like $m = 40$ will not occur everywhere throughout the supply network. Indeed, and more generally, one might ask whether the regularity in the production tree is responsible for the sharpness of the transition. To investigate this possibility, in Figure 2 we plot² $\rho_T(x)$ for a supply tree with irregular complexity, where different tiers may have different values of m_t . Here we construct 4 trees whose complexity increases with T . The first tree has $T = 4$ and $m_2 = 2$, $m_3 = 6$ and $m_4 = 10$. The second tree has $T = 7$, $m_t = 2$ for $t = 2, 3$, $m_t = 6$, for $t = 4, 5$ and $m_t = 10$, for $t = 6, 7$. The third tree has $T = 10$, $m_t = 2$ for $t = 2, 3, 4$, $m_t = 6$, for $t = 5, 6, 7$ and $m_t = 10$, for $t = 8, 9, 10$. Finally, the fourth tree has large length (here $T = 1000$), $m_t = 2$ for t below the first tercile, $m_t = 6$ for t between the first and the second terciles and $m_t = 10$ for t above the second tercile. We see that trees of moderate length once again exhibit a sharp transition in their probability of successful production. This feature is thus not at all dependent upon the regularity of the trees.

²In Fig 2, all curves are plotted on a grid $x = [0, 1]$ with increments of 0.001. The black curve represents $\rho_T(x)$ for $T = 1000$ and the discontinuity marks the first time $\rho_T(x)$ is positive. At a finer grid, this figure would be continuous for any large T , but would converge pointwise to the shape plotted as $T \rightarrow \infty$.

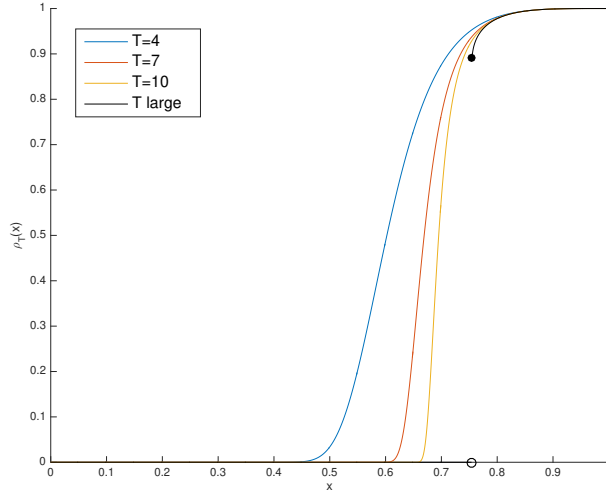


FIGURE 2. Successful production probability $\rho_T(x)$ for different finite numbers of tiers T , but where different tiers may have different complexity m .

SA1.2. Convergence of the reliability function to that of the basic model. Now fix m and n as in our main model, but focus on a supply tree with T tiers. Let $\rho_T(x)$ denote the probability of successful production when every firm requiring inputs invests x . A first observation is that $\rho_T(x)$ and its derivative converge uniformly to the function $\rho(x)$ from the basic model everywhere except near x_{crit} .

LEMMA SA1.

- (1) For all $T \geq 1$, the function $x \mapsto \rho_T(x)$ defined for $x \in (0, 1)$ is strictly increasing and infinitely differentiable.
- (2) On any compact set excluding x_{crit} , the sequence $(\rho_T(x))_{T=1}^{\infty}$ converges uniformly to $\rho(x)$ and $(\rho'_T(x))_{T=1}^{\infty}$ converges uniformly to $\rho'(x)$.

Proof. The sequence $(\rho_T)_{T=1}^{\infty}$ is a sequence of monotonically increasing³ and differentiable functions, converging pointwise to ρ . We know that ρ is continuous and has finite derivative on any compact set excluding x_{crit} . Therefore, by Dini's theorem, the functions ρ_T converge uniformly to ρ ; the analogous statement holds with derivatives. \square

SA1.3. Equilibrium. This section makes the point that our principal findings about equilibrium investment extend to the T -tier model. We first specify the economic environment in which firms will make their decisions, which is a variant of the one studied in the main text. The purpose of this model is to make each production process finite almost surely. With such a process replacing the mechanics of our baseline production network, we redo our main exercises. We characterize the analogue of the reliability function ρ , showing that it has the qualitative properties documented in the previous subsection. We then show the main result: that for κ in an intermediate range, all equilibria are fragile. This shows that infinite production chains were not essential to our main findings.

³Either by considering the \mathcal{R} introduced in Section 2.3 or by a simple coupling argument, increasing x must increase $\rho_T(x)$.

For a tractable finite-tier model, we use production processes with a length T that is geometrically distributed; it turns out this allows us to have both a consistent description of production as well as one-dimensional fixed point equations characterizing the outcomes.

Associated with each product i there is now a continuum of *processes* π , $\pi \in [0, 1]$, which are simultaneously occurring throughout the network. Each firm if , whenever it is in process $\pi \in [0, 1]$ now draws a $T_{if\pi}$ for that process, which is called the *depth* of the process. The $T_{if\pi}$ are geometrically distributed, with an expectation τ . Thus, depths are almost surely finite. The firm does not know the realization of these depths when it chooses x_{if} . Note that the geometric distribution ensures consistency in the following sense: if a firm requires inputs for a certain process (i.e., if $T_{if\pi} \geq 1$) then each of its suppliers jf' have depths $T_{jf'\pi}$ that are also geometrically distributed with expectation τ . Let $\tilde{\rho}_\tau(x)$ denote the probability of successful production when every firm has relationship strength x . Let $\tilde{\rho}_\tau(x_{if}, x)$ denote the probability of successful production by firm if when every other firm has relationship strength x and firm if invests x_{if} .

We define the $\tau = \infty$ game to be the one studied in the main model; it is one where $T = \infty$ with probability 1.

We take $\underline{x} = 0$ for simplicity. Conditional on entering and making an investment x_{if} , the net expected profit of firm if is, as before

$$\Pi_{if}(x_{if}; x, \bar{f}) = \underbrace{\kappa g(\bar{f} \tilde{\rho}_\tau(x))}_{G(\bar{f} \tilde{\rho}_\tau(x))} \underbrace{\tilde{\rho}_\tau(x_{if}, x)}_{\mathbf{E}[F_{if}]} - c(x_{if}) - \Phi(f),$$

An equilibrium requires that all entering firms make non-negative profits, no non-entering firm could make positive profits by entering and that all entering firms are choosing investments to maximize their profits.

The first order condition for investment, equating marginal benefits and marginal costs of increasing x_{if} , is

$$\kappa g(\bar{f} \tilde{\rho}_\tau(x)) \frac{d}{dx_{if}} \tilde{\rho}_\tau(x, x_{if}^*) = c'(x_{if}^*). \quad (\text{SA-1})$$

To make this more explicit, let $\nu = 1 - 1/\tau$ be the success/continuation probability of the geometric random variable with expectation τ . We may write

$$\begin{aligned} \tilde{\rho}_\tau(x, x_{if}) &= \sum_{T=0}^{\infty} [(1-\nu)^T \nu] [1 - (1 - x_{if} \rho_{T-1}(x))^n]^m \\ \frac{d}{dx_{if}} \tilde{\rho}_\tau(x, x_{if}) &= \sum_{T=0}^{\infty} [(1-\nu)^T \nu] \frac{d}{dx_{if}} [1 - (1 - x_{if} \rho_{T-1}(x))^n]^m \\ &= mn\nu \sum_{T=0}^{\infty} [(1-\nu)^T] \rho_{T-1}(x) (1 - x_{if} \rho_{T-1}(x))^{n-1} [1 - (1 - x_{if} \rho_{T-1}(x))^n]^{M-1} \end{aligned}$$

Now if we require that the first-order condition holds for everyone at a symmetric investment equilibrium, we get

$$\kappa g(\bar{f} \tilde{\rho}_\tau(x^*)) mn\nu \sum_{T=0}^{\infty} [(1-\nu)^T] \rho_{T-1}(x^*) (1 - x^* \rho_{T-1}(x^*))^{n-1} [1 - (1 - \rho_{T-1}(x^*))^n]^{M-1} = c'(x^*). \quad (\text{SA-2})$$

We define an equilibrium just as in Definition 1. For a given entry level $\bar{\gamma}$ and corresponding symmetric investment equilibrium x^* , a *free-entry* symmetric equilibrium satisfies

$$\Phi(\bar{f}) = G(\bar{f}\tilde{\rho}_\tau(x^*))\tilde{\rho}_\tau(x^*) - c(x^*). \quad (\text{SA-3})$$

PROPOSITION SA1. Recall Theorem 1. There is a $\tilde{\kappa} \in (\underline{\kappa}, \bar{\kappa})$ so that for each $\kappa \in (\tilde{\kappa}, \bar{\kappa})$, if τ is sufficiently small, there is a positive investment equilibrium (of the τ -economy) with x arbitrarily close to x_{crit} .

Proof. Consider $\kappa = \bar{\kappa}$. Take any sequence $(\tau_k)_k$ converging to zero. It is straightforward to show that an equilibrium with positive effort exists for the τ -economy with τ sufficiently small.⁴ All entering firms must be making non-negative net profits, while investing a strictly positive amount in robustness. As gross profits must cover these costs of investment in robustness, both x_k^* and $\tilde{\rho}_\tau(x_k^*)$ must be bounded away from 0. Passing to a subsequence, we may assume they converge.

Suppose, toward a contradiction, that x_k^* converges to any value other than x_{crit} . Then we may restrict attention to a compact set excluding x_{crit} . By the uniform convergence established in the Lemma SA1, the limit gives an equilibrium of the $\tau = 0$ game. But this is a contradiction to the assumption about the $\tau = 0$ game.

Now, if $\kappa = \bar{\kappa} - \delta$ for a small enough δ , the existence proof in the footnote still goes through. Thus there is a compact interval of positive measure where positive-investment equilibria exist. By compactness, the convergence of x to x_{crit} can be made uniform. \square

It follows, (using the convergence established in the lemma) that an arbitrarily small shock to \underline{x} results in the collapse of production. Formally, for any $\epsilon > 0$ there exists a $\bar{\tau}$ sufficiently large such that for all $\tau \geq \bar{\tau}$ the equilibrium $x^*(\tau)$ is fragile for a shock to \underline{x} of size ϵ .

Thus, the main result of the paper extends to the τ -economy: there is an open set of economies where very small shocks lead to collapse in production.

Moreover, for a given r and \bar{f} , by choosing κ and Φ suitably, one can always ensure that (\bar{f}, r) constitute an equilibrium satisfying the zero-profit condition.⁵ Thus, in this model, fragility is compatible with the marginal firm being just indifferent to entering.

SA1.3.1. Concluding remarks. It should be noted that this modified economy can feature either a zero-profit equilibrium or one where firms make positive profits.

SA2. HETEROGENEITY

In this appendix, we first explain how we numerically solve the examples from Section 6, and then report some additional information about these examples.

⁴The proof of Proposition 3 extends to show that for entry sufficiently low, positive investment occurs. As entry increases, the same proof also shows that investment and profits decrease. By Assumption 4, for sufficient entry profits would be negative for large enough τ . Thus, some interior level of entry involves both a positive level of investment and satisfies the no-profitable-entry condition. Throughout these arguments, the uniform convergence of $\tilde{\rho}_\nu(\tau)$ and its derivatives to those of the basic model shows that all the properties we need extend.

⁵First solve (SA-2) for κ , and then take any Φ satisfying (SA-3) at the desired \bar{f} .

SA2.1. Solving examples with heterogeneities numerically. To compute the equilibria in Examples 1 and 2, we proceed as follows.

As before we consider symmetric equilibria, in the sense that: (i) all entering producers of product i invest the same amount sourcing a given input j , and (ii) entry is described by a threshold on entry costs, which is now product-specific (\bar{f}_i). As we are interested in these symmetric investment choices conditional on entering, and about the entry decision of the marginal producer of product i , we dispense with the subscript f when we consider investments. In such an equilibrium the profit of a marginal producer of product i is given by

$$\Pi_{i,\bar{f}_i} = G_i(\bar{f}_i r_i) r_i - \frac{1}{2} \sum_{j \in \mathcal{I}_i} \gamma_{ij} x_{ij}^2 - \Phi_i(\bar{f}_i), \quad (\text{SA-4})$$

where

$$r_i = \prod_{j \in \mathcal{I}_i} (1 - (1 - x_{ij} r_j)^{n_{ij}}),$$

\mathcal{I}_i is the neighborhood of i on the product dependency graph, with $|\mathcal{I}_i| = m_i$ (the complexity of production for product i), and n_{ij} is the number of potential suppliers a producer of product i has for input j (i.e. the potential level of multisourcing by producers of product i for input j). Note also that in equation (SA-4), the second term represents the effort cost function and we have set $\underline{x}_{ij} = 0$ for simplicity.

The marginal benefit a producer of product i receives from investing in its relationships with suppliers of input j is

$$MB_{ij} = G(\bar{f}_i r_i) \prod_{l \in \mathcal{I}_i, l \neq j} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{ij} (1 - x_{ij} r_j)^{n_{ij}-1} r_j. \quad (\text{SA-5})$$

Letting $\gamma_{ij} = 1$ (as in the examples), the marginal cost for a producer of product i investing in a relationship with a supplier of input j is

$$MC_{ij} = x_{ij}. \quad (\text{SA-6})$$

We look for $|\mathcal{I}| \times |\mathcal{I}|$ matrix X , with entries x_{ij} satisfying $MB_{ij} = MC_{ij}$. The value of x_{i1} that equates the marginal benefits and marginal costs for firm i 's investment into sourcing product 1 is increasing in $G_i(\bar{f}_i r_i) = \alpha_i (1 - r_i \bar{f}_i)$, which an entering producer of product i takes a given. As we still have the freedom to choose α_i we can select an arbitrary $x_{i1} \in (0, 1)$. However, doing so pins down the value of x_{ij} for all $j \neq 1$. Specifically, we must have

$$\frac{MB_{ij}}{MB_{i1}} = \frac{MC_{ij}}{MC_{i1}}, \quad \forall i, j$$

which can be expressed as

$$\frac{G(\bar{f}_i r_i) \prod_{l \in \mathcal{I}_i, l \neq j} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{ij} (1 - x_{ij} r_j)^{n_{ij}-1} r_j}{G(\bar{f}_i r_i) \prod_{l \in \mathcal{I}_i, l \neq 1} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{i1} (1 - x_{i1} r_1)^{n_{i1}-1} r_1} = \frac{x_{ij}}{x_{i1}},$$

and reduces to

$$\frac{(1 - x_{ij}r_j)^{n_{ij}-1}}{(1 - (1 - x_{ij}r_j)^{n_{ij}})} \frac{n_{ij}}{x_{ij}} = \frac{n_{i1} r_1}{x_{i1} r_j} \frac{(1 - x_{i1}r_1)^{n_{i1}-1}}{(1 - (1 - x_{i1}r_1)^{n_{i1}})}. \quad (\text{SA-7})$$

The left-hand side is decreasing in x_{ij} while the right-hand side is given, so there can be only one solution x_{ij} satisfying the above.

We initialize x_{i1} for all i to values just smaller than 1. We then decrease these values incrementally by a small amount. After each reduction we calculate the X matrix using the above procedure and calculate the probability of successful production r_i for each industry. We continue until the probability of successful production decreases to 0 for one of the products i . This gives us values of X such that at least one product is in the critical regime.

The values of $G_i(r_i \bar{f}_i)$ are then set so that $MB_{i1} = MC_{i1}$ using equations (SA-5) and (SA-6). This ensures that all firms are choosing profit maximizing investments that result in at least one product being fragile. Recall that $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$, and so depends on both α_i and \bar{f}_i . Thus for a given value of $G_i(r_i \bar{f}_i)$ we can choose an arbitrary $\bar{f}_i \in (0, 1)$ and then set α_i so that $G_i(r_i \bar{f}_i)$ is the value we require. The values of \bar{f}_i we pick don't matter because entry costs can be set to ensure that all entering firms make weakly positive profits, and no positive measure of non-entering firms want to enter. Specifically, using the entry cost function, $\Phi_i(f) = \beta_i f$, we choose the values of β_i that set the profit of the marginal firm to 0 (in equation (SA-4)) for the products where production is not critical. Likewise, we choose values of β_i that set this profit to a strictly positive number for the products where production is in the critical regime.

Using this procedure there turn out to be essentially two types of equilibria with fragile firms. Either a firm in the set $\{a, b, c, d\}$ becomes fragile first, in which case all firms become fragile simultaneously, or else a firm in the set $\{e, f, g\}$ becomes fragile first in which case all firms in this set simultaneously become fragile while firms in the set $\{a, b, c, d\}$ do not. When a firm in the set $\{a, b, c, d\}$ becomes fragile first, a shock to any one of $\{a, b, c, d\}$ that reduces the reliability of sourcing an input (either directly, or indirectly by reducing incentives to invest in reliability) is sufficient for the probability of successful production of all firms to fall to 0. When a firm in the set $\{e, f, g\}$ becomes fragile first, a similar shock to any one of these firms is sufficient for the probability of successful production of these firms, but not firms $\{a, b, c, d\}$, to fall to 0. The parameters selected in Examples 1 and 2 are chosen to illustrate these two possible cases.

SA2.2. Example 1—additional information. The equilibrium relationship strengths are reported in the matrix X below, where an entry x_{ij} represents the strength chosen by a producer of product i in a relationship sourcing input j .

By pinning down the first column of X with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.8873 & 0.8872 & 0.9315 & 0.9385 & 0 & 0 & 0 \\ 0.8773 & 0 & 0.9204 & 0.9272 & 0 & 0 & 0 \\ 0.8673 & 0.8672 & 0 & 0.9084 & 0 & 0 & 0 \\ 0.8573 & 0.8572 & 0.8915 & 0 & 0 & 0 & 0 \\ 0.7573 & 0 & 0 & 0 & 0 & 0.9726 & 0.9783 \\ 0.7473 & 0 & 0 & 0 & 0.9464 & 0 & 0.9572 \\ 0.7373 & 0 & 0 & 0 & 0.9265 & 0.9317 & 0 \end{bmatrix}.$$

Such an X corresponds to the following product reliabilities:

$$r = [0.9926, 0.9928, 0.9387, 0.9307, 0.5384, 0.5262, 0.5145].$$

Also, for such values of x_{ij} , products a, b, c, d are non critical, while products e, f, g are critical.

We can obtain $G = [21.0836, 17.7538, 3.2818, 3.0451, 2.6780, 2.5859, 2.4990]$.

Recall that we let $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$. Setting

$$\alpha = [40, 30, 15, 10, 3.5, 3, 2.8],$$

we can find the values

$$\bar{f} = [0.4764, 0.4112, 0.8322, 0.7473, 0.4362, 0.2623, 0.2089].$$

Given those entry fractions \bar{f}_i , we then choose values of β_i that set the profit of the marginal firm to 0 for products a, b, c, d and values that set this profit to a strictly positive number for products e, f, g . Such values are

$$\beta = [40.4397, 39.8544, 2.3021, 2.2770, 0.3000, 0.4000, 0.5000].$$

Gross profits *before* the entry costs are

$$\tilde{\Pi} = [19.2666, 16.3872, 1.9159, 1.7016, 0.2036, 0.1756, 0.1506],$$

and the net profits of the marginal entering producer of each product (after entry costs) are

$$\Pi = [0, 0, 0, 0, 0.0728, 0.0707, 0.0461].$$

SA2.3. Example 2—additional information. The equilibrium investment levels are reported in the matrix X below, where an entry x_{ij} represents the investment made by a producer of product i towards sourcing input j .

By pinning down the first column of X with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.7965 & 0.7792 & 0.8735 & 0.8663 & 0 & 0 & 0 \\ 0.8065 & 0 & 0.8859 & 0.8785 & 0 & 0 & 0 \\ 0.8165 & 0.8029 & 0 & 0.8681 & 0 & 0 & 0 \\ 0.8265 & 0.8124 & 0.8855 & 0 & 0 & 0 & 0 \\ 0.8965 & 0 & 0 & 0 & 0 & 0.8947 & 0.8894 \\ 0.9065 & 0 & 0 & 0 & 0.9103 & 0 & 0.8992 \\ 0.9165 & 0 & 0 & 0 & 0.9204 & 0.9146 & 0 \end{bmatrix}.$$

Such an X corresponds to the following product reliabilities:

$$r = [0.8837, 0.9132, 0.7653, 0.7756, 0.8778, 0.8865, 0.8951].$$

Also, for such values of x_{ij} , the production of products a, b, c, d is critical, while the production of products e, f, g are now non critical.

We can obtain $G = [3.7758, 3.9399, 1.9995, 2.0736, 2.6608, 2.7929, 2.9372]$.

Recall that we let $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$. Setting

$$\alpha = [4, 5, 6, 7, 10, 15, 20],$$

we can find the values

$$\bar{f} = [0.0634, 0.2322, 0.8712, 0.9074, 0.8361, 0.9180, 0.9531].$$

Given those entry fractions \bar{f}_i , we then choose values of β_i that set the profit of the marginal firm to a strictly positive number for products a, b, c, d , and values that set the profits of the marginal entering firm to 0 for products e, f, g . Such values are

$$\beta = [10, 4, 0.2, 0.2, 1.3613, 1.3580, 1.4345].$$

Gross profits *before* the entry costs are

$$\tilde{\Pi} = [1.9590, 2.4942, 0.4978, 0.5446, 1.1381, 1.2466, 1.3673],$$

and the net profits of the marginal entering producer of each product (after entry costs) are

$$\Pi = [1.3246, 1.5655, 0.3236, 0.3632, 0, 0, 0].$$

SA3. OMITTED PROOFS

SA3.1. **Lemma 2.** Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (\text{SA-8})$$

Then there are values $x_{\text{crit}}, r_{\text{crit}} \in (0, 1]$ such that:

- (i) $\rho(x) = 0$ for all $x < x_{\text{crit}}$;
- (ii) ρ has a (unique) point of discontinuity at x_{crit} ;
- (iii) ρ is strictly increasing for $x \geq x_{\text{crit}}$;
- (iv) the inverse of ρ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[r_{\text{crit}}, 1]$, where $r_{\text{crit}} = \rho(x_{\text{crit}})$;
- (v) χ is positive and quasiconvex on the domain $(0, 1]$;
- (vi) $\chi'(r_{\text{crit}}) = 0$.

Proof. We first list some properties of ρ and χ .

Property 0: For positive r in the range of ρ , we have $r = \rho(x)$ if and only if

$$x = \frac{1 - (1 - r^{1/m})^{1/n}}{r}.$$

This is shown by rearranging equation PC in the paper.

Property 1: $\chi(1) = 1$. This follows by inspection.

Property 2: $\chi(r) > 0$ for all $r \in (0, 1]$. This follows by inspection.

Property 3: $\lim_{r \downarrow 0} \chi(r) = \infty$. This follows by an application of l'Hopital's rule, i.e.

$$\lim_{r \downarrow 0} \frac{\frac{d}{dr} \left(1 - (1 - r^{1/m})^{1/n}\right)}{\frac{d}{dr} r} = \lim_{r \downarrow 0} \frac{(1 - r^{1/m})^{1/n-1} r^{1/m-1}}{m n} = \infty.$$

Property 4: $\lim_{r \uparrow 1} \chi'(r) = \infty$. This follows by examining

$$\chi'(r) = \frac{r^{1/m}(1 - r^{1/m})^{1/n-1}}{mnr^2} + \frac{(1 - r^{1/m})^{1/n} - 1}{r^2}$$

Property 5: There is a unique interior $r_{\text{crit}} \in (0, 1)$ minimizing $\chi(r)$. To show this, define $z(r) = 1/(1 - r^{1/m})$, and note that $r \in (0, 1)$ satisfies $\chi'(r) = 0$ if and only if the corresponding $z(r) > 1$ solves

$$z - 1 = mn(z^{1/n} - 1);$$

here we use that the function z is a bijection from $(0, 1)$ to $(1, \infty)$. The equation clearly has exactly one solution for $z > 1$.⁶ Now it remains to see that the unique $r \in (0, 1)$ solving $\chi'(r) = 0$ defines a local minimum. Note that χ is continuously differentiable on $(0, 1)$. Property 3 implies that $\chi'(r) < 0$ for some $r < r_{\text{crit}}$ while Property 4 implies that $\chi'(r) > 0$ for some $r > r_{\text{crit}}$. These points suffice to show Property 5.

To prove the lemma, we relate desired properties of ρ to properties of χ ; the claims made here can be visualized by referring to Figure 18 in the paper, panels (A) and (C), which illustrate the properties of the functions involved.

Together, Properties 0 and 5 imply that there is no $r > 0$ in the range of ρ such that $x < \chi(r_{\text{crit}})$. Let $x_{\text{crit}} = \chi(r_{\text{crit}})$, which yields (vi) of the lemma; then what we have said implies $\rho(x) = 0$ for $x < x_{\text{crit}}$, i.e., statement (i) of the lemma. The proof of Property 5 also implies statement (v) in the Lemma.

It remains to show (ii-iv) of the Lemma. By definition, $\rho(x)$ is the largest solution of (PC). Properties 1 and 5 imply that on the domain $[r_{\text{crit}}, 1]$, χ is a strictly increasing function whose range is $[x_{\text{crit}}, 1]$. Fix an $r \in [r_{\text{crit}}, 1]$ and let $x = \chi(r)$. What we have said implies that (x, r) solves (PC) and that there is no r' with $r' > r$ such that (x, r') solves (PC). Thus, by definition $r = \rho(x)$. Notice that as we vary r in the interval, x varies over the interval $[x_{\text{crit}}, 1]$. This establishes (iv), and (iii) follows immediately from the fact that χ is increasing on the domain in question. For (ii), it suffices to deduce from Properties 2 and 5 that the minimum of χ has both coordinates positive.

We now consider the case $n = 1$. Here, $\frac{d\chi(r)}{dr} = \frac{r^{1/m} - mr^{1/m}}{mr^2} < 0$ for all $r \in (0, 1]$. Since from Properties 1 and 3 (which still hold when $n = 1$), $\chi(1) = 1$ and $\lim_{r \downarrow 0} \chi(r) = \infty$, it follows that $\chi(r)$ is decreasing in r and has image $[1, \infty)$. Thus, by logic similar to the above, $\rho(x) = 0$ for all $x \in [0, 1)$ and $\rho(x) = 1$ when $x = 1$. It follows that $x_{\text{crit}} = 1$ in this case and all the statements of the lemma are satisfied, though some of them are trivial. \square

SA3.2. Lemma 3. Fix any $m \geq 2$, $n \geq 2$, and $x \geq x_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n , and x) such $0 \leq x_1 < x_2 < 1/\rho(x)$ so that:

0. $Q(0; x) = Q(1/\rho(x); x) = 0$ and $Q(x_{if}; x) > 0$ for all $x_{if} \in (0, 1/\rho(x))$;
1. $Q(x_{if}; x)$ is increasing and convex in x_{if} on an interval $[0, x_1]$;
2. $Q(x_{if}; x)$ is increasing and concave in x_{if} on an interval $(x_1, x_2]$;
3. $Q(x_{if}; x)$ is decreasing in x_{if} on an interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

⁶The left-hand side is linear and the right-hand side is concave, since $n \geq 2$. At $z = 1$ the two sides are equal, and the curves defined by the left-hand and right-hand sides are not tangent, so there is exactly one solution $z > 1$.

Proof. As a piece of notation, define

$$\zeta(x_{if}; x) = 1 - x_{if}\rho(x).$$

When using ζ , we will often omit the arguments for brevity. Then

$$Q(x_{if}; x) = mn\rho(x)\zeta^{n-1}(1 - \zeta^n)^{M-1},$$

from which it follows immediately that $Q(0; x) = Q(1/\rho(x); x) = 0$ and $Q(x_{if}; x) > 0$ for all $x_{if} \in (0, 1/\rho(x))$. This establishes Property 0 in the lemma statement.

Next, we can calculate

$$Q'(x_{if}; x) = -mn\rho(x)^2\zeta^{n-2}(1 - \zeta^n)^{M-2}[(mn - 1)\zeta^n - n + 1].$$

Note that for $x_{if} \in (0, 1/\rho(x))$ we have⁷

$$\text{sign}[Q'(x_{if}, x)] = \text{sign}[(mn - 1)\zeta^n - n + 1]. \quad (\text{SA-9})$$

Further, for sufficiently small $x_{if} > 0$

$$\text{sign}[(mn - 1)\zeta^n - n + 1] = \text{sign}[n(m - 1)] > 0.$$

Thus $Q'(0; x) > 0$.

We will now deduce from the above calculations about Q' that there is exactly one local maximum of $x_{if} \mapsto Q(x_{if}; x)$ on its domain, $[0, 1/\rho(x)]$. First, as this is a continuous function with $Q(0; x) = 0 = Q(1/\rho(x); x)$ and $Q'(0; x) > 0$, it follows there is an interior maximum of $Q(x_{if}; x)$ in the interval $(0, 1/\rho(x))$. Next, by (SA-9), $\text{sign}[Q'(x_{if}, x)] > 0$ if and only if

$$x_{if} < \frac{1 - \left(\frac{n-1}{mn-1}\right)^{\frac{1}{n}}}{\rho(x)}.$$

Thus, there can be at most one value of x_{if}^* with $Q'(x_{if}^*; x) = 0$. Together, these observations imply that Q has one local optimum on its extended domain, which is in fact a global maximum. We let x_2 be defined by the unique value of x_{if} at which $Q'(x_{if}^*; x) = 0$. This establishes Property 3. It also establishes the ‘‘increasing’’ part of Properties 1 and 2, since $Q(x_{if}; x)$ is increasing to the left of x_2 by what we have said.

The next part of the proof studies the second derivative of Q to establish the claims about the convexity/concavity of Q . First note that

$$Q''(x_{if}; x) = mn\rho(x)^3\zeta^{n-3}(1 - \zeta^n)^{M-3}H$$

where

$$H = \underbrace{(m^2n^2 - 3mn + 2)}_A \zeta^{2n} + \underbrace{((1 - 3m)n^2 + (3m + 3)n - 4)}_B \zeta^n + \underbrace{n^2 - 3n + 2}_C.$$

For $x_{if} \in (0, 1/\rho(x))$, we can see that

$$\text{sign}[Q''(x_{if}; x)] = \text{sign}[H]. \quad (\text{SA-10})$$

Let $z := \zeta^n$ for $z \in (0, 1)$ (which corresponds to $x_{if} \in (0, 1/\rho(x))$). We can then write $H = \tilde{H}(z)$ for $z \in (0, 1)$, where

$$\tilde{H}(z) = Az^2 + Bz + C, \quad (\text{SA-11})$$

⁷The sign operator is +1 for positive numbers, -1 for negative numbers, and 0 when the argument is 0.

and A , B and C are constants (labeled above) depending only on m, n . \tilde{H} is therefore a quadratic polynomial in z and its roots depend only on n and m . Further, $A > 0$, $B < 0$, $C > 0$ and $A + B + C > 0$. Thus \tilde{H} is convex in z with $\tilde{H}(0) > 0$ and $\tilde{H}(1) > 0$.

We first argue that $\min_{z \in [0,1]} \tilde{H}(z) < 0$. Towards a contradiction suppose $\min_{z \in [0,1]} \tilde{H}(z) \geq 0$. This implies that Q'' is nonnegative by equation SA-10 and hence that Q is globally convex. However, we have already established that $Q'(0; x) > 0$, so the convexity of Q implies there can be no interior maximum which contradicts our deductions above.

An immediate implication of $\min_{z \in [0,1]} \tilde{H}(z) < 0$ is that $\tilde{H}(z)$ has two real roots, z_1 and $z_2 < z_1$. This establishes the basic shape of $\tilde{H}(z)$ as illustrated in Figure 3.

It will be helpful to sometimes consider the values of x_{if} that correspond to the roots of the $\tilde{H}(z)$. To this end, we define the function

$$X(z) := \frac{1 - z^{1/n}}{\rho(x)}. \quad (\text{SA-12})$$

We can then set $x_1 = X(z_1)$ (i.e., the first inflection point of Q). Along with what we already know, the deduced shape of $\tilde{H}(z)$ pins down the remaining properties we require about the shape of Q , as we now argue. For an illustration, see Figure 3.

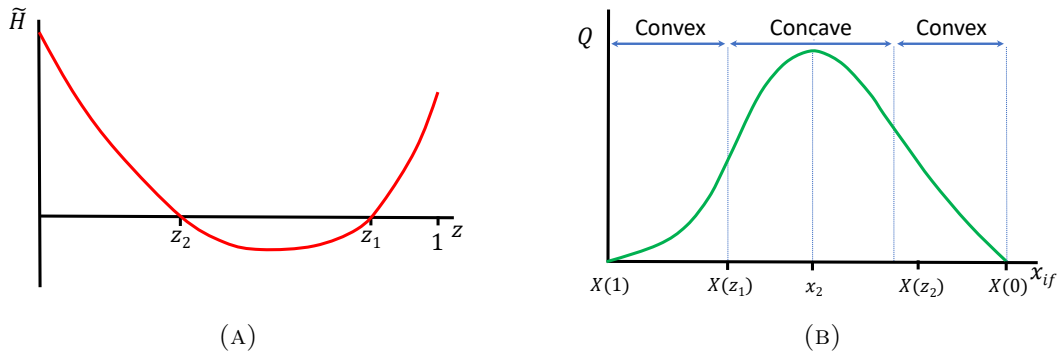


FIGURE 3. Panel (a) shows basic shape of the function $\tilde{H}(z)$. Panel(b) shows the basic shape of the function $Q(x_{if}; x)$, where the convexity and concavity in different regions is implied by the shape of $\tilde{H}(z)$.

As $z_1 > z_2$, $\tilde{H}'(z_1) > 0$. This corresponds to $Q''(x_{if}; x)$ going from positive to negative as x_{if} crosses $x_1 := X(z_1)$, and thus $Q(x_{if}; x)$ going from convex to concave. As $Q'(0; x) > 0$ and $Q(x_{if}; x)$ is convex for $x_{if} \in [0, x_1]$, the maximum of Q must occur at a value of $x_{if} \in (x_1, 1)$. Recalling that we let x_2 be the value of x_{if} at which $Q(x_{if}; x)$ is maximized we conclude that $x_1 < x_2$. Further, as for values of $x_{if} < x_1$ the function $Q(x_{if}; x)$ is convex we have established the convexity part of Property 2 of Lemma 3.

By similar reasoning, $\tilde{H}'(z_2) < 0$. This corresponds to $Q''(x_{if}; x)$ going from negative to positive as x_{if} crosses $X(z_2)$. Recall that x_2 is defined as the maximum of Q . We must have $X(z_2) > x_2$. If not, Q would be increasing and convex for all $x_{if} \geq X(z_2)$, contradicting the existence of an interior maximum as already established. This establishes that the function Q remains concave until after its maximum point x_2 establishing the concavity part of Property 3.

We now argue that $x_1 < x_{\text{crit}}$ to establish Property 4. We do so in two steps. First we show it for the case in which firms other than if are investing at the critical level, i.e., $x = x_{\text{crit}}$, and then for $x > x_{\text{crit}}$.

First then, we need to establish that

$$x_1 = \frac{1 - z_1^{1/n}}{\rho(x_{\text{crit}})} < \frac{1 - (1 - \rho(x_{\text{crit}})^{1/m})^{1/n}}{\rho(x_{\text{crit}})} = x_{\text{crit}}.$$

Writing $\rho(x_{\text{crit}})$ as r_{crit} , this holds if and only if $z > 1 - r_{\text{crit}}^{1/m}$, which is equivalent to $r_{\text{crit}} > (1 - z)^m$. A sufficient condition for this to hold is that $\chi'((1 - z)^m) < 0$, since we know from Lemma 2 in the paper that $\chi'(r_{\text{crit}}) = 0$.

Note that

$$\chi'(r) = \frac{\frac{r^{1/m}(1-r^{1/m})^{1/n-1}}{mn} + (1 - r^{1/m})^{1/n} - 1}{r^2}$$

Putting $r = (1 - z)^m$ in the above yields

$$\chi'((1 - z)^m) = \frac{\frac{(1-z)z^{1/n-1}}{mn} + (z)^{1/n} - 1}{(1 - z)^{2m}}.$$

The denominator is always positive, hence we only need to verify that the numerator is negative, when evaluated at the larger of the two roots of \tilde{H} , which we will call z_1 . The numerator simplifies to

$$\text{Num}(z) = z^{1/n} \left(\frac{1 - z}{mnz} + 1 \right) - 1$$

and must now be evaluated at the root z_1 . The latter is given by the quadratic formula as

$$z_1 = \frac{D + \sqrt{D^2 - 4(n^2 - 3n + 2)(m^2n^2 - 3mn + 2)}}{2(m^2n^2 - 3mn + 2)},$$

where $D = 3mn^2 - 3mn - n^2 - 3n + 4$. This expression for z_1 simplifies to

$$z_1 = \frac{n \left(3m(n - 1) + \sqrt{(m - 1)(n - 1)(5mn - m - n - 7)} - n - 3 \right) + 4}{2(mn - 2)(mn - 1)}.$$

Now note that the shape of $\text{Num}(z)$ is pictured in Fig. 4:

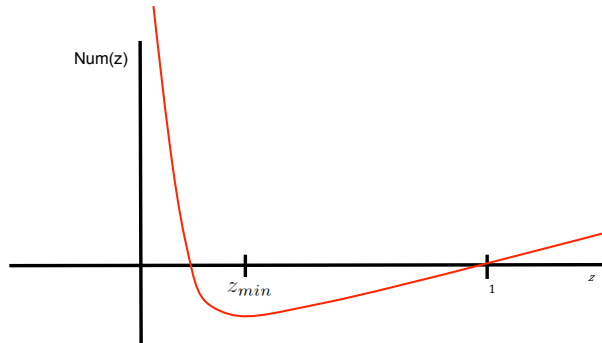


FIGURE 4. Shape of $\text{Num}(z)$.

Thus a sufficient condition for $\text{Num}(z_1) < 0$ is that $z_1 > z_{\text{min}}$.

It is easy to find $z_{min} = \frac{n-1}{mn-1}$ by solving $\frac{dNum(z)}{dz} = 0$. Indeed,

$$\frac{dNum(z)}{dz} = \frac{z^{1/n-2}(z(mn-1) + 1 - n)}{mn^2}$$

and thus a first-order condition is satisfied when $z(mn-1) + 1 - n = 0$.

We also note that since $z^{1/n-2}$ and mn^2 are always positive,

$$\text{sign} \left[\frac{dNum(z)}{dz} \right] = \text{sign} [z(mn-1) + 1 - n],$$

and thus the function $Num(z)$ is decreasing on $z \in [0, z_{min})$ and increasing on $z \in (z_{min}, \infty)$.

Since $Num(1) = 0$, the above imply that $Num(z) < 0$ over the range $[z_{min}, 1)$. Thus, we only need to check that $z_1 > z_{min}$ in order to guarantee that $Num(z_1) < 0$.

It is easy to verify that the following inequality holds:

$$\begin{aligned} z_{min} &= \frac{n-1}{mn-1} \\ &< \frac{n \left(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3 \right) + 4}{2(mn-2)(mn-1)} = z_1. \end{aligned}$$

Indeed, multiplying both sides by $2(mn-2)(mn-1)$ yields

$$2(mn-2)(n-1) < n \left(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3 \right) + 4$$

which, after some algebra, reduces to

$$0 < n(mn+1-m-n) + n\sqrt{(m-1)(n-1)(5mn-m-n-7)}.$$

Since all terms on the right-hand side are clearly positive when $m, n \geq 2$, the inequality holds and we conclude that $z_1 > 1 - r_{crit}^{1/m}$. This establishes that $\hat{x} < x_{crit}$ when $x = x_{crit}$.

We now consider $x > x_{crit}$. From Eq. (SA-12), it is clear that x_1 is decreasing in $\rho(x)$. Since from Lemma 2 in the paper $\rho(x)$ is increasing in x , for $x \geq x_{crit}$, it follows that x_1 is decreasing in x . This establishes Property 4 of Lemma 3, completing the proof of Lemma 3. □

SA3.3. Lemma 5. The function $\beta(r)$ is quasiconcave and has a maximum at $\hat{r} := \left(\frac{(2m-1)n}{2mn-1} \right)^m$.

Proof. Recall that $\beta(r) = mnr^{2-\frac{1}{m}}(1-r^{\frac{1}{m}})^{1-\frac{1}{n}}$. We will show that $\beta(r)$ is quasiconcave by demonstrating that there exists an $\hat{r} \in (0, 1)$ such that $\beta'(r) > 0$ for $r \in (0, \hat{r})$, $\beta'(r) < 0$ for $r \in (\hat{r}, 1)$ and $\beta'(r) = 0$ for $r = \hat{r}$.

$$\beta'(r) = mn \left(2 - \frac{1}{m} \right) r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}} \right)^{1-\frac{1}{n}} - \left(1 - \frac{1}{n} \right) mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}} \right)^{-\frac{1}{n}} \left(\frac{1}{m} r^{\frac{1}{m}-1} \right)$$

$$= mn r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left[\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) \frac{1}{m} \right] \right].$$

Note that the the first factor, $mn r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}}$, exceeds 0 for any $r \in (0, 1)$ and is equal to 0 for $r = 0, 1$. Moreover, the expression multiplying it,

$$\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) \left(\frac{1}{m}\right) \right],$$

is a strictly decreasing, continuous function of r . It is also strictly positive when $r = 0$ and strictly negative when $r = 1$, implying that it is equal to 0 at some $r \in (0, 1)$. It follows that there exists an $\hat{r} \in (0, 1)$ satisfying the claimed properties.

Having established that $\beta(r)$ is quasiconcave and $\beta'(\hat{r}) = 0$, it follows immediately that $\beta(r)$ is maximized at $r = \hat{r}$. To find \hat{r} , we use its defining property and solve the following equation:

$$r \left((2m - 1) n r^{-1/m} - 2mn + 1 \right) \left(1 - r^{1/m}\right)^{-1/n} = 0.$$

As $r \left(1 - r^{1/m}\right)^{-1/n} > 0$ for any positive production equilibrium, the equation is solved by

$$\hat{r} = \left(\frac{(2m - 1)n}{2mn - 1} \right)^m.$$

□

SA3.4. Lemma 6. Recall that $\hat{r} := \left(\frac{(2m-1)n}{2mn-1} \right)^m$. For all $n \geq 2$ and $m \geq 3$, $\hat{r} < r_{\text{crit}}$.

Proof. By Lemma 2 in the paper, the function $\chi(r)$ is positive and quasiconvex on the domain $[0, 1]$ with $r_{\text{crit}} = \operatorname{argmin}_r \chi(r)$. Thus, $\hat{r} < r_{\text{crit}}$ if and only if $\chi'(\hat{r}) < 0$.

Recall that $\chi(r) = \frac{1 - (1 - r^{1/m})^{1/n}}{r}$ and so

$$\chi'(r) = \frac{\frac{1}{n} (1 - r^{1/m})^{1/n-1} \frac{1}{m} r^{1/m} - (1 - (1 - r^{1/m})^{1/n})}{r^2}.$$

We will study this expression evaluated at $\hat{r} = \left(\frac{(2m-1)n}{2mn-1} \right)^m$ from Lemma 5 in the paper.

The denominator of the above expression for $\chi'(r)$ is always positive, so we need only check that the numerator is negative. Calling $A = \frac{n-1}{2mn-1}$ and $B = \frac{n+1-1/m}{n-1}$, we may rewrite the numerator, after some simplifications, as $A^{1/n} B - 1$, and thus we need only check that

$$A^{1/n} B < 1 \tag{SA-13}$$

Let $h(n, m) = A^{1/n} B$. To demonstrate (SA-13), we will show that:

- Step 1: $h(n, 3) < 1$, for all $n \geq 2$.
- Step 2: $h(n, m)$ is decreasing in m , for all $n \geq 2$.

Step 1: First note that

$$h(n, 3) = \left(\frac{n-1}{6n-1} \right)^{1/n} \left(\frac{n+1-1/3}{n-1} \right)$$

We will show that

- Step 1a: $h(n, 3)$ is increasing in n .

- Step 1b: $\lim_{n \rightarrow \infty} h(n, 3) = 1$.

From this we can conclude that $h(n, 3) < 1$, for all $n \geq 2$.

To show Step 1a, note that

$$\frac{\partial h(n, 3)}{\partial n} = -\frac{\left(\frac{n-1}{6n-1}\right)^{1/n} \left((18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n\right)}{3n^2 \frac{(n-1)}{(6n-1)}}$$

The denominator is positive for all $n \geq 2$. Looking at the numerator, since $\left(\frac{n-1}{6n-1}\right)^{1/n} > 0$ for all $n \geq 2$, it suffices to show that

$$(18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n < 0 \text{ for all } n \geq 2$$

in order to ensure that $\frac{\partial h(n, 3)}{\partial n} > 0$. It is easy to show that

$$\ln\left(\frac{n-1}{6n-1}\right) < \ln\left(\frac{1}{6}\right) < -1.79$$

Thus, for all $n \geq 2$

$$\begin{aligned} (18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n &< -(18n^2 + 9n - 2)1.79 + 30n^2 + 10n \\ &< -32n^2 - 16n + 4 + 30n^2 + 10n \\ &= -2n^2 - 6n + 4 \\ &< 0. \end{aligned}$$

We thus conclude⁸ that $h(n, 3)$ is increasing in n and Step 1a is proved.

Step 1b follows immediately by noting that

$$\lim_{n \rightarrow \infty} h(n, 3) = \lim_{n \rightarrow \infty} \left(\frac{1}{6}\right)^{1/n} = 1$$

We have thus proved Step 1.

Step 2. To show that $h(n, m)$ is decreasing in m , let us note that, for all $n \geq 2$

$$\begin{aligned} \frac{\partial h(n, m)}{\partial m} &= B \frac{\partial A^{1/n}}{\partial m} + A^{1/n} \frac{\partial B}{\partial m} \\ &= \left(\frac{n-1}{2mn-1}\right)^{1/n} \left(\frac{-2}{m(n-1)} \frac{mn(1+1/n)-1}{2mn-1} + \frac{1}{(n-1)m^2}\right) \\ &< \left(\frac{n-1}{2mn-1}\right)^{1/n} \left(\frac{-1}{m(n-1)} + \frac{1}{(n-1)m^2}\right) \\ &< 0. \end{aligned}$$

where the second equality follows after some simplifications, while the first inequality follows from the fact that $\frac{mn(1+1/n)-1}{2mn-1} > \frac{1}{2}$, which is easy to check.

We have thus shown that $h(n, m)$ is decreasing in m , for any $n \geq 2$, and Step 2 is thus proved.

This concludes the proof of the lemma. \square

⁸It is worth noting that this argument does not work for $m = 2$, in which case the numerator of $\frac{\partial h(n, 2)}{\partial n}$ could be negative and thus $h(n, 2)$ could be decreasing.

SA3.5. **Lemma 7.** The function $x^*(\bar{f})$ is decreasing in \bar{f} on any interval of values of \bar{f} where $x^*(\bar{f}) > x_{\text{crit}}$. Moreover, $G(r\bar{f})$ is also decreasing in \bar{f} on an interval of values of \bar{f} where r is positive.

Proof. In this proof, we take r , x , g , and a new function we define called h all to be implicit functions of \bar{f} at the unique positive investment equilibrium. To save on notation we take $\underline{x} = 0$ in the proof.

Consider the following system of equations in the variables r , x , g , h , and x :

$$x = \chi(r) \tag{SA-14}$$

$$g = g(r\bar{f}) \tag{SA-15}$$

$$h = r^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} \tag{SA-16}$$

$$kgh = c'(x) \tag{SA-17}$$

where k is the constant $k = \kappa mn$. The first equation is, by Lemma 2 from the paper, equivalent to physical consistency positive investment equilibrium. The second is the definition of g .⁹ The third is a definition of an auxiliary symbol h that allows us to write the final equation parsimoniously, which is the condition that marginal benefits are equal to marginal cost at an equilibrium.

At a positive investment equilibrium, this system of equations is satisfied by some values of the variables. Considering an interval of values of \bar{f} where a positive investment equilibrium exists, we will implicitly differentiate this system to study the comparative statics of the equilibrium, which is valid because the equations are necessary conditions at the unique positive investment equilibrium. Write a dot over an endogenous variable to denote its derivative in \bar{f} . We make several observations following from the equations displayed above.

1. \dot{r} and \dot{x} have the same sign,¹⁰ by equation (SA-14) and Lemma 2 in the paper.
2. \dot{h} has a sign opposite¹¹ of \dot{r} (and \dot{x}) because by equation (SA-16) and Lemma 5 in the paper, h is a decreasing function of r for a positive investment equilibrium.
3. $\dot{g} = g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]$, using the chain rule on $g = g(r\bar{f})$.
4. Differentiating $kgh = c'(x)$ implicitly and using point 3, we have

$$\begin{aligned} k(g\dot{h} + \dot{g}h) &= c''(x)\dot{x} \\ k(g\dot{h} + [g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]]) &= c''(x)\dot{x}. \end{aligned}$$

Now, suppose toward a contradiction that $\dot{x} \geq 0$. The right-hand side is then weakly positive because c'' is positive. The $g\dot{h}$ term is weakly negative by point 2. The $[g'(r\bar{f}) \cdot [r + \dot{r}\bar{f}]]$ is strictly negative because $r > 0$ and $\dot{r} \geq 0$. This is a contradiction.

Now to show that $G(r\bar{f})$ is also decreasing in \bar{f} , for all values of \bar{f} where r is positive, simply note from point 4 that

$$k\dot{g}h = c''(x)\dot{x} - k\dot{g}h$$

⁹We have abused notation by using g for the function value as well as the function.

¹⁰That is, one is positive if and only if the other is positive, and if one is zero then the other is.

¹¹If one is positive, the other is negative. If one is zero, so is the other.

where $c''(x)\dot{x} < 0$ and $kgh > 0$ (from point 2). Thus the right-hand side is negative. This then implies that $\dot{g} < 0$, since k and h are positive. The result thus immediately follows. \square

SA3.6. **Lemma 8.** H has the following properties:

- $H(0) > 0$;
- $H(1) < 1$;
- $H(\bar{f})$ is strictly decreasing for all \bar{f} such that $x^*(\bar{f}) > 0$, and $H(\bar{f}) = 0$ for all \bar{f} such that $x^*(\bar{f}) = 0$.

Proof. Recall the definition

$$H(\bar{f}) = \Phi^{-1}(\max\{G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})), 0\}).$$

Suppose the entry cutoff \bar{f} is in effect and $x^*(\bar{f}) \geq x_{\text{crit}}$ is the positive level of investment. We show that the profits any firm if makes by entering the market are decreasing in \bar{f} . To this end, consider two values \bar{f}', \bar{f}'' such that $\bar{f}'' < \bar{f}'$, $x^*(\bar{f}) \geq x_{\text{crit}}$ when \bar{f} is either of the two. For profits upon entering and investing x_{if} to be non-negative for firm if given an entry level \bar{f} , it must be the case that

$$G(\bar{f}\rho(x^*(\bar{f}))) (1 - (1 - x_{if}\rho(x^*(\bar{f})))^n)^m - c(x_{if}) - \Phi(f) \geq 0. \quad (\text{SA-18})$$

By Lemma 7 in the paper, when $x^*(\bar{f})$ is positive, both $\rho(x^*(\bar{f}))$ and $G(\bar{f}\rho(x^*(\bar{f})))$ are decreasing in \bar{f} . Thus, after a reduction in entry from \bar{f}' to \bar{f}'' , firm if can always deviate from the new equilibrium investment level by continuing to choose $x_{if} = x^*(\bar{f}')$ —the optimal investment level prior to the reduction. With this choice, the positive profit constraint given by inequality (SA-18) becomes slacker. Re-optimizing its investment choice, firm if 's profits from entering can only weakly increase causing inequality (SA-18) to become weakly slacker still. Thus equilibrium profits, before entry costs, are strictly decreasing in \bar{f} when they are non-negative. As Φ is strictly increasing in its argument, so is Φ^{-1} . Thus when $x^*(\bar{f}) > 0$, $H(\bar{f})$ is strictly decreasing in \bar{f} .

As there exists a positive and symmetric equilibrium by assumption, and equilibrium profits are decreasing in \bar{f} when they are non-negative, we must have $G(0)\rho(x^*(0)) - c(x^*(0)) > 0$. As $\Phi(0) = 0$ and Φ is a strictly increasing function, this proves that $H(0) > 0$. By Assumption 4 in the paper, $H(1) < 1$.

We have shown that $H(\bar{f})$ is strictly decreasing in \bar{f} when $x^*(\bar{f}) > 0$. We show now that if $x^*(\bar{f}) = 0$ then $H(\bar{f}) = 0$. This follows immediately from the definition of $H(\bar{f})$ because $\rho(0) = 0$. \square