SUPPLEMENTARY APPENDIX
SUPPLY NETWORK FORMATION AND FRAGILITY

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This document contains supporting material for the paper "Supply Network Formation and Fragility," which herein we refer to as the "main paper" or simply "paper."

SA1. Omitted Proofs

In this section we restate the results in the paper for which proofs were omitted, and then provide the missing proofs.

SA1.1. **Lemma 2.** Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \tag{SA-1}$$

Then there are values $x_{\text{crit}}, \overline{r}_{\text{crit}} \in (0, 1]$ such that:

(i) $\widehat{\rho}(x) = 0$ for all $x < x_{\text{crit}}$;
(ii) $\widehat{\rho}$ has a (unique) point of discontinuity at x_{crit};
(iii) $\widehat{\rho}$ is strictly increasing for $x \geq x_{\text{crit}}$;
(iv) the inverse of $\widehat{\rho}$ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[\overline{r}_{\text{crit}}, 1]$, where $\overline{r}_{\text{crit}} = \widehat{\rho}(x_{\text{crit}})$;
(v) χ is positive and quasiconvex on the domain $(0, 1]$;
(vi) $\chi'(\overline{r}_{\text{crit}}) = 0$.

Proof. We first list some properties of $\widehat{\rho}$ and χ.

Property 0: For positive r in the range of $\widehat{\rho}$, we have $r = \widehat{\rho}(x)$ if and only if

$$x = \frac{1 - (1 - r^{1/m})^{1/n}}{r}.$$

This is shown by rearranging equation (6) in the paper.

Property 1: $\chi(1) = 1$. This follows by inspection.

Property 2: $\chi(r) > 0$ for all $r \in (0, 1]$. This follows by inspection.

Property 3: $\lim_{r \downarrow 0} \chi(r) = \infty$. This follows by an application of l'Hopital's rule, i.e.

$$\lim_{r \downarrow 0} \frac{\frac{d}{dr}\left(1 - (1 - r^{1/m})^{1/n}\right)}{\frac{d}{dr} r} = \lim_{r \downarrow 0} \frac{(1 - r^{1/m})^{1/n - 1} r^{1/m - 1}}{mn} = \infty.$$

Property 4: $\lim_{r \uparrow 1} \chi'(r) = \infty$. This follows by examining

$$\chi'(r) = \frac{r^{1/m}(1 - r^{1/m})^{1/n - 1}}{mnr^2} + \frac{(1 - r^{1/m})^{1/n} - 1}{r^2}$$

Property 5: There is a unique interior $\overline{r}_{\text{crit}} \in (0, 1)$ minimizing $\chi(r)$. To show this, define $z(r) = 1/(1 - r^{1/m})$, and note that $r \in (0, 1)$ satisfies $\chi'(r) = 0$ if and only if the corresponding $z(r) > 1$ solves

$$z - 1 = mn(z^{1/n} - 1);$$

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here we use that the function z is a bijection from $(0,1)$ to $(1,\infty)$. The equation clearly has exactly one solution for $z > 1$.[1] Now it remains to see that the unique $r \in (0,1)$ solving $\chi'(r) = 0$ defines a local minimum. Note that χ is continuously differentiable on $(0,1)$. Property 3 implies that $\chi'(r) < 0$ for some $r < \overline{r}_{\text{crit}}$ while Property 4 implies that $\chi'(r) > 0$ for some $r > \overline{r}_{\text{crit}}$. These points suffice to show Property 5.

To prove the lemma, we relate desired properties of $\widehat{\rho}$ to properties of χ; the claims made here can be visualized by referring to Figure 11 in the paper, panels (A) and (B), which illustrate the properties of the functions involved.

Together, Properties 0 and 5 imply that there is no $r > 0$ in the range of $\widehat{\rho}$ such that $x < \chi(\overline{r}_{\text{crit}})$. Let $x_{\text{crit}} = \chi(\overline{r}_{\text{crit}})$, which yields (vi) of the lemma; then what we have said implies $\widehat{\rho}(x) = 0$ for $x < x_{\text{crit}}$, i.e., statement (i) of the lemma. The proof of Property 5 also implies statement (v) in the Lemma.

It remains to show (ii-iv) of the Lemma. By definition, $\widehat{\rho}(x)$ is the largest solution of (6) in the paper. Properties 1 and 5 imply that on the domain $[\overline{r}_{\text{crit}}, 1]$, χ is a strictly increasing function whose range is $[x_{\text{crit}}, 1]$. Fix an $r \in [\overline{r}_{\text{crit}}, 1]$ and let $x = \chi(r)$. What we have said implies that (x, r) solves (6) in the paper and that there is no r' with $r' > r$ such that (x, r') solves (6) in the paper. Thus, by definition $r = \widehat{\rho}(x)$. Notice that as we vary r in the interval, x varies over the interval $[x_{\text{crit}}, 1]$. This establishes (iv), and (iii) follows immediately from the fact that χ is increasing on the domain in question. For (ii), it suffices to deduce from Properties 2 and 5 that the minimum of χ has both coordinates positive.

We now consider the case $n = 1$. Here, $\frac{d\chi(r)}{dr} = \frac{r^{1/m} - mr^{1/m}}{mr^2} < 0$ for all $r \in (0,1]$. Since from Properties 1 and 3 (which still hold when $n = 1$), $\chi(1) = 1$ and $\lim_{r \downarrow 0} \chi(r) = \infty$, it follows that $\chi(r)$ is decreasing in r and has image $[1,\infty)$. Thus, by logic similar to the above, $\widehat{\rho}(x) = 0$ for all $x \in [0,1)$ and $\widehat{\rho}(x) = 1$ when $x = 1$. It follows that $x_{\text{crit}} = 1$ in this case and all the statements of the lemma are satisfied, though some of them are trivial. □

SA1.2. **Lemma 4.** Fix any $m \geq 2$, $n \geq 2$, and $r \geq \underline{r}_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n, and x) such $0 \leq x_1 < x_2 < 1/r$ and so that:

0. $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$;
1. $Q(x_{if}; r)$ is increasing and convex in x_{if} on the interval $[0, x_1]$;
2. $Q(x_{if}; r)$ is increasing and concave in x_{if} on the interval $(x_1, x_2]$;
3. $Q(x_{if}; r)$ is decreasing in x_{if} on the interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

Proof. As a piece of notation, define

$$\zeta(x_{if}; r) = 1 - x_{if} r.$$

When using ζ, we will often omit the arguments for brevity. Then

$$Q(x_{if}; r) = mnr\zeta^{n-1}(1 - \zeta^n)^{M-1}.$$

Statement 0. It follows immediately from this equation that $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$. This establishes Claim 0 in the lemma statement.

Statements 1–3. Establishing Statements 1–3 is more involved; we begin by studying the first derivative of Q to establish the increasing/decreasing statements, and then move to the second derivative to establish the convex/concave statements.

We can calculate

$$Q'(x_{if}; r) = -mnr^2\zeta^{n-2}\left(1 - \zeta^n\right)^{M-2}\left[(mn - 1)\,\zeta^n - n + 1\right].$$

Note that for $x_{if} \in (0, 1/r)$ we have[2]

$$\text{sign}\left[Q'(x_{if}; r)\right] = \text{sign}\left[(mn - 1)\,\zeta^n - n + 1\right]. \tag{SA-2}$$

Further, for sufficiently small $x_{if} > 0$

$$\text{sign}\left[(mn - 1)\,\zeta^n - n + 1\right] = \text{sign}\left[n(m - 1)\right] > 0.$$

[1]The left-hand side is linear and the right-hand side is concave, since $n \geq 2$. At $z = 1$ the two sides are equal, and the curves defined by the left-hand and right-hand sides are not tangent, so there is exactly one solution $z > 1$.

[2]The sign operator is $+1$ for positive numbers, -1 for negative numbers, and 0 when the argument is 0.

Thus $Q'(0;r) > 0$.

We will now deduce from the above calculations about Q' that there is exactly one local maximum of $x_{if} \mapsto Q(x_{if};r)$ on its domain, $[0,1/r]$. First, as this is a continuous function with $Q(0;r) = 0 = Q(1/r;r)$ and $Q'(0;r) > 0$, it follows there is an interior maximum of $Q(x_{if};r)$ in the interval $(0,1/r)$. Next, by (SA-2), $\text{sign}\left[Q'(x_{if};r)\right] > 0$ if and only if

$$x_{if} < \frac{1 - \left(\frac{n-1}{mn-1}\right)^{\frac{1}{n}}}{r}.$$

Thus, there can be at most one value of x^*_{if} with $Q'(x^*_{if};r) = 0$. Together, these observations imply that Q has one local optimum on its extended domain, which is in fact a global maximum. We let x_2 be defined by the unique value of x_{if} at which $Q'(x^*_{if};r) = 0$. This establishes Property 3. It also establishes the "increasing" part of Properties 1 and 2, since $Q(x_{if};r)$ is increasing to the left of x_2 by what we have said.

The next part of the proof studies the second derivative of Q to establish the claims about the convexity/concavity of Q. First note that

$$Q''(x_{if};r) = mnr^3\zeta^{n-3}\left(1-\zeta^n\right)^{M-3}H$$

where

$$H = \underbrace{\left(m^2n^2 - 3mn + 2\right)}_{A}\zeta^{2n} + \underbrace{\left(\left(1-3m\right)n^2 + \left(3m+3\right)n - 4\right)}_{B}\zeta^n + \underbrace{n^2 - 3n + 2}_{C}.$$

For $x_{if} \in (0,1/r)$, we can see that

$$\text{sign}\left[Q''(x_{if};r)\right] = \text{sign}\left[H\right]. \tag{SA-3}$$

Let $z := \zeta^n$ for $z \in (0,1)$ (which corresponds to $x_{if} \in (0,1/r)$). We can then write $H = \widetilde{H}(z)$ for $z \in (0,1)$, where

$$\widetilde{H}(z) = Az^2 + Bz + C, \tag{SA-4}$$

and A, B and C are constants (labeled above) depending only on m,n. $\widetilde{H}$ is therefore a quadratic polynomial in z and its roots depend only on n and m. Further, $A > 0$, $B < 0$, $C > 0$ and $A+B+C > 0$. Thus $\widetilde{H}$ is convex in z with $\widetilde{H}(0) > 0$ and $\widetilde{H}(1) > 0$.

We first argue that $\min_{z\in[0,1]}\widetilde{H}(z) < 0$. Towards a contradiction suppose $\min_{z\in[0,1]}\widetilde{H}(z) \geq 0$. This implies that Q'' is nonnegative by equation SA-3 and hence that Q is globally convex. However, we have already established that $Q'(0;r) > 0$, so the convexity of Q implies there can be no interior maximum, which contradicts our deductions above.

An immediate implication of $\min_{z\in[0,1]}\widetilde{H}(z) < 0$ is that $\widetilde{H}(z)$ has two real roots, z_1 and $z_2 < z_1$. This establishes the basic shape of $\widetilde{H}(z)$ as illustrated in Figure 1.

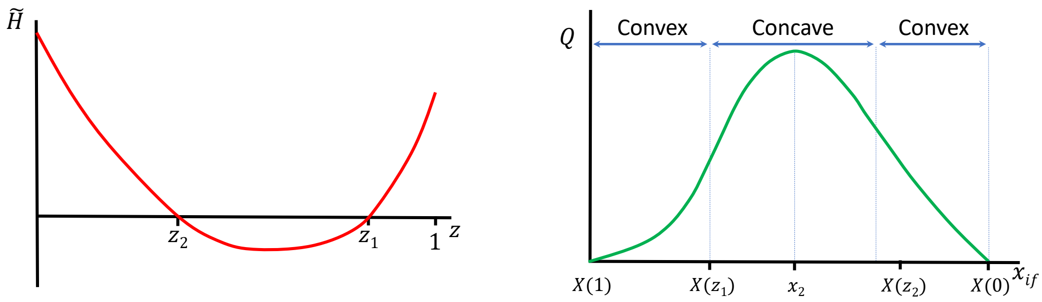
It will be helpful to sometimes consider the values of x_{if} that correspond to the roots of the $\widetilde{H}(z)$. To this end, we define the function

$$X(z) := \frac{1 - z^{1/n}}{r}. \tag{SA-5}$$

We can then set $x_1 = X(z_1)$ (i.e., the first inflection point of Q). Along with what we already know, the deduced shape of $\widetilde{H}(z)$ pins down the remaining properties we require about the shape of Q, as we now argue. For an illustration, see Figure 1.

As $z_1 > z_2$, it follows that $\widetilde{H}'(z_1) > 0$. This corresponds to $Q''(x_{if};r)$ going from positive to negative as x_{if} crosses $x_1 := X(z_1)$, and thus $Q(x_{if};r)$ going from convex to concave. As $Q'(0;r) > 0$ and $Q(x_{if};r)$ is convex for $x_{if} \in [0,x_1]$, the maximum of Q must occur at a value of $x_{if} \in (x_1,1)$. Recalling that we let x_2 be the value of x_{if} at which $Q(x_{if};r)$ is maximized we conclude that $x_1 < x_2$. Further, as for values of $x_{if} < x_1$ the function $Q(x_{if};r)$ is convex, we have established the convexity part of Property 2 of Lemma 4.

By similar reasoning, $\widetilde{H}'(z_2) < 0$. This corresponds to $Q''(x_{if};r)$ going from negative to positive as x_{if} crosses $X(z_2)$. Recall that x_2 is defined as the maximum of Q. We must have $X(z_2) > x_2$. If not, Q would be increasing and convex for all $x_{if} \geq X(z_2)$, contradicting the existence of an interior maximum as already established. This establishes that the function Q remains concave until after its maximum



(A) (B)

FIGURE 1. Panel (a) shows basic shape of the function $\widetilde{H}(z)$. Panel(b) shows the basic shape of the function $Q(x_{if};r)$, where the convexity and concavity in different regions is implied by the shape of $\widetilde{H}(z)$.

point x_2 establishing the concavity part of Property 3. This completes our demonstration of Statements 1–3.

Statement 4. We now argue that $x_1 < x_{\text{crit}}$ to establish Statement 4. We do so in two steps. First we show it for the case in which firms other than if have reliability $r = \overline{r}_{\text{crit}}$, and deduce the $r > \overline{r}_{\text{crit}}$ case.

The case $r = \overline{r}_{\text{crit}}$. We need to establish that

$$x_1 = \frac{1 - z_1^{1/n}}{\overline{r}_{\text{crit}}} < \frac{1 - (1 - \overline{r}_{\text{crit}}^{1/m})^{1/n}}{\overline{r}_{\text{crit}}} = x_{\text{crit}}.$$

This holds if and only if $z > 1 - \overline{r}_{\text{crit}}^{1/m}$, which is equivalent to $\overline{r}_{\text{crit}} > (1-z)^m$. A sufficient condition for this to hold is that $\chi'((1-z)^m) < 0$, since we know from Lemma 2 in the paper that $\chi'(\overline{r}_{\text{crit}}) = 0$.

Note that

$$\chi'(r) = \frac{\frac{r^{1/m}(1-r^{1/m})^{1/n-1}}{mn} + (1 - r^{1/m})^{1/n} - 1}{r^2}$$

Setting $r = (1-z)^m$ in the above yields

$$\chi'((1-z)^m) = \frac{\frac{(1-z)z^{1/n-1}}{mn} + (z)^{1/n} - 1}{(1-z)^{2m}}.$$

The denominator is always positive, hence we only need to verify that the numerator is negative, when evaluated at the larger of the two roots of $\widetilde{H}$, which we will call z_1. The numerator simplifies to

$$\text{Num}(z) = z^{1/n}\Big(\frac{1-z}{mnz} + 1\Big) - 1$$

and must now be evaluated at the root z_1. This root is given by the quadratic formula as

$$z_1 = \frac{D + \sqrt{D^2 - 4(n^2 - 3n + 2)(m^2n^2 - 3mn + 2)}}{2(m^2n^2 - 3mn + 2)},$$

where $D = 3mn^2 - 3mn - n^2 - 3n + 4$. This expression for z_1 simplifies to

$$z_1 = \frac{n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn - m - n - 7)} - n - 3\Big) + 4}{2(mn-2)(mn-1)}.$$

Now we claim that the shape of $\text{Num}(z)$ is pictured in Fig. 2:

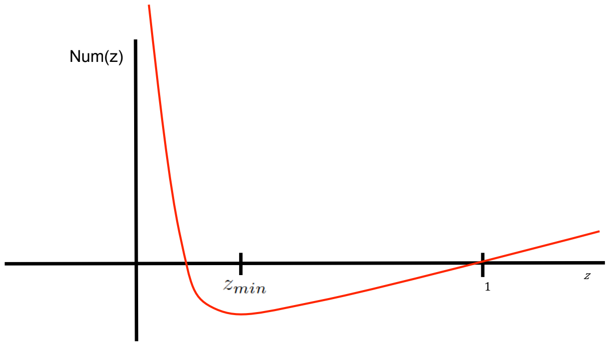


FIGURE 2. Shape of Num(z).

Thus a sufficient condition for $\text{Num}(z_1) < 0$ is that $z_1 > z_{min}$.

We will now calculate z_{min} and verify the shape of $\text{Num}(z)$ is as claimed. It is easy to show that $z_{min} = \frac{n-1}{mn-1}$ by solving $\frac{d\text{Num}(z)}{dz} = 0$.[3] We also note that since $z^{1/n-2}$ and mn^2 are always positive,

$$\text{sign}\left[\frac{d\text{Num}(z)}{dz}\right] = \text{sign}\left[z(mn-1)+1-n\right],$$

and thus the function $\text{Num}(z)$ is decreasing on $z \in [0, z_{min})$ and increasing on $z \in (z_{min}, \infty)$.

Since $\text{Num}(1) = 0$, the above imply that $\text{Num}(z) < 0$ over the range $[z_{min}, 1)$. Thus, we only need to check that $z_1 > z_{min}$ in order to guarantee that $\text{Num}(z_1) < 0$.

It is easy to verify that the following inequality holds[4]:

$$\begin{aligned} z_{min} &= \frac{n-1}{mn-1} \\ &< \frac{n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\Big) + 4}{2(mn-2)(mn-1)} = z_1. \end{aligned}$$

We conclude (recalling the beginning of this proof) that $z_1 > 1 - \overline{r}_{\text{crit}}^{1/m}$. This establishes that $\widehat{x} < x_{\text{crit}}$ when $r = \overline{r}_{\text{crit}}$.

The case $r > \overline{r}_{\text{crit}}$. From Eq. (SA-5), it is clear that x_1 is decreasing in r. This establishes Property 4 of Lemma 4, completing the proof of Lemma 4. □

SA1.3. **Lemma 8.** The function $\beta(r)$ is quasiconcave and has a maximum at $\widehat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Proof. Recall that $\beta(r) = mnr^{2-\frac{1}{m}}(1-r^{\frac{1}{m}})^{1-\frac{1}{n}}$. We will show that $\beta(r)$ is quasiconcave by demonstrating that there exists an $\widehat{r} \in (0,1)$ such that $\beta'(r) > 0$ for $r \in (0, \widehat{r})$, $\beta'(r) < 0$ for $r \in (\widehat{r}, 1)$ and $\beta'(r) = 0$ for $r = \widehat{r}$.

$$\begin{aligned} \beta'(r) &= mn\left(2 - \frac{1}{m}\right) r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} - \left(1 - \frac{1}{n}\right) mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left(\frac{1}{m} r^{\frac{1}{m}-1}\right) \\ &= mnr^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left[\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) \frac{1}{m}\right]\right]. \end{aligned}$$

[3] Indeed, $\frac{d\text{Num}(z)}{dz} = \frac{z^{1/n-2}(z(mn-1)+1-n)}{mn^2}$ and thus a first-order condition is satisfied when $z(mn-1)+1-n = 0$.

[4] Indeed, multiplying both sides by $2(mn-2)(mn-1)$ yields

$$2(mn-2)(n-1) < n\Big(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\Big) + 4$$

which, after some algebra, reduces to

$$0 < n(mn+1-m-n) + n\sqrt{(m-1)(n-1)(5mn-m-n-7)}.$$

Since all terms on the right-hand side are clearly positive when $m, n \geq 2$, the inequality holds.

Note that the the first factor, $mnr^{1-\frac{1}{m}}\left(1-r^{\frac{1}{m}}\right)^{-\frac{1}{n}}$, exceeds 0 for any $r \in (0,1)$ and is equal to 0 for $r = 0, 1$. Moreover, the expression multiplying it,

$$\left(2-\frac{1}{m}\right)-r^{\frac{1}{m}}\left[\left(2-\frac{1}{m}\right)+\left(1-\frac{1}{n}\right)\left(\frac{1}{m}\right)\right],$$

is a strictly decreasing, continuous function of r. It is also strictly positive when $r = 0$ and strictly negative when $r = 1$, implying that it is equal to 0 at some $r \in (0,1)$. It follows that there exists an $\widehat{r} \in (0,1)$ satisfying the claimed properties.

Having established that $\beta(r)$ is quasiconcave and $\beta'(\widehat{r}) = 0$, it follows immediately that $\beta(r)$ is maximized at $r = \widehat{r}$. To find $\widehat{r}$, we use its defining property and solve the following equation:

$$r\left((2m-1)nr^{-1/m}-2mn+1\right)\left(1-r^{1/m}\right)^{-1/n}=0.$$

As $r\left(1-r^{1/m}\right)^{-1/n} > 0$ for any positive production equilibrium, the equation is solved by

$$\widehat{r}=\left(\frac{(2m-1)n}{2mn-1}\right)^{m}.$$

This completes the proof. □

SA1.4. **Lemma 9.** Recall that $\widehat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$. For all $n \geq 2$ and $m \geq 3$, $\widehat{r} < \overline{r}_{\text{crit}}$.

Proof. By Lemma 2 in the paper, the function $\chi(r)$ is positive and quasiconvex on the domain $[0,1]$ with $\overline{r}_{\text{crit}} = \operatorname{argmin}_r \chi(r)$. Thus, $\widehat{r} < \overline{r}_{\text{crit}}$ if and only if $\chi'(\widehat{r}) < 0$.

Recall that $\chi(r) = \frac{1-(1-r^{1/m})^{1/n}}{r}$ and so

$$\chi'(r)=\frac{\frac{1}{n}(1-r^{1/m})^{1/n-1}\frac{1}{m}r^{1/m}-\left(1-(1-r^{1/m})^{1/n}\right)}{r^2}.$$

We will study this expression evaluated at $\widehat{r} = \left(\frac{(2m-1)n}{2mn-1}\right)^m$ from Lemma 8 in the paper.

The denominator of the above expression for $\chi'(r)$ is always positive, so we need only check that the numerator is negative. Calling $A = \frac{n-1}{2mn-1}$ and $B = \frac{n+1-1/m}{n-1}$, we may rewrite the numerator, after some simplifications, as $A^{1/n}B - 1$, and thus we need only check that

$$A^{1/n}B<1 \tag{SA-6}$$

Let $h(n,m) = A^{1/n}B$ .To demonstrate (SA-6), we will show that:

- Step 1: $h(n,3) < 1$, for all $n \geq 2$.
- Step 2: $h(n,m)$ is decreasing in m, for all $n \geq 2$.

Step 1: First note that

$$h(n,3)=\left(\frac{n-1}{6n-1}\right)^{1/n}\left(\frac{n+1-1/3}{n-1}\right)$$

We will show that

- Step 1a: $h(n,3)$ is increasing in n.
- Step 1b: $\lim_{n\to\infty} h(n,3) = 1$.

From this we can conclude that $h(n,3) < 1$, for all $n \geq 2$.

To show Step 1a, note that

$$\frac{\partial h(n,3)}{\partial n}=-\frac{\left(\frac{n-1}{6n-1}\right)^{1/n}\left((18n^2+9n-2)\ln(\frac{n-1}{6n-1})+30n^2+10n\right)}{3n^2\frac{(n-1)}{(6n-1)}}$$

The denominator is positive for all $n \geq 2$. Looking at the numerator, since $\left(\frac{n-1}{6n-1}\right)^{1/n} > 0$ for all $n \geq 2$, it suffices to show that

$$(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right)+30n^2+10n<0 \quad \text{for all } n\geq 2$$

in order to ensure that $\frac{\partial h(n,3)}{\partial n} > 0$. It is easy to show that

$$\ln\left(\frac{n-1}{6n-1}\right) < \ln\left(\frac{1}{6}\right) < -1.79$$

Thus, for all $n \geq 2$

$$\begin{aligned}(18n^2+9n-2)\ln\left(\frac{n-1}{6n-1}\right)+30n^2+10n &< -(18n^2+9n-2)1.79+30n^2+10n\\ &< -32n^2-16n+4+30n^2+10n\\ &= -2n^2-6n+4\\ &< 0.\end{aligned}$$

We thus conclude[5] that $h(n,3)$ is increasing in n and Step 1a is proved.

Step 1b follows immediately by noting that

$$\lim_{n\to\infty} h(n,3) = \lim_{n\to\infty}\left(\frac{1}{6}\right)^{1/n} = 1$$

We have thus proved Step 1.

Step 2. To show that $h(n,m)$ is decreasing in m, let us note that, for all $n \geq 2$

$$\begin{aligned}\frac{\partial h(n,m)}{\partial m} &= B\frac{\partial A^{1/n}}{\partial m}+A^{1/n}\frac{\partial B}{\partial m}\\ &= \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-2}{m(n-1)}\frac{mn(1+1/n)-1}{2mn-1}+\frac{1}{(n-1)m^2}\right)\\ &< \left(\frac{n-1}{2mn-1}\right)^{1/n}\left(\frac{-1}{m(n-1)}+\frac{1}{(n-1)m^2}\right)\\ &< 0.\end{aligned}$$

where the second equality follows after some simplifications, while the first inequality follows from the fact that $\frac{mn(1+1/n)-1}{2mn-1} > \frac{1}{2}$, which is easy to check.

We have thus shown that $h(n,m)$ is decreasing in m, for any $n \geq 2$, and Step 2 is thus proved.

This concludes the proof of the lemma. □

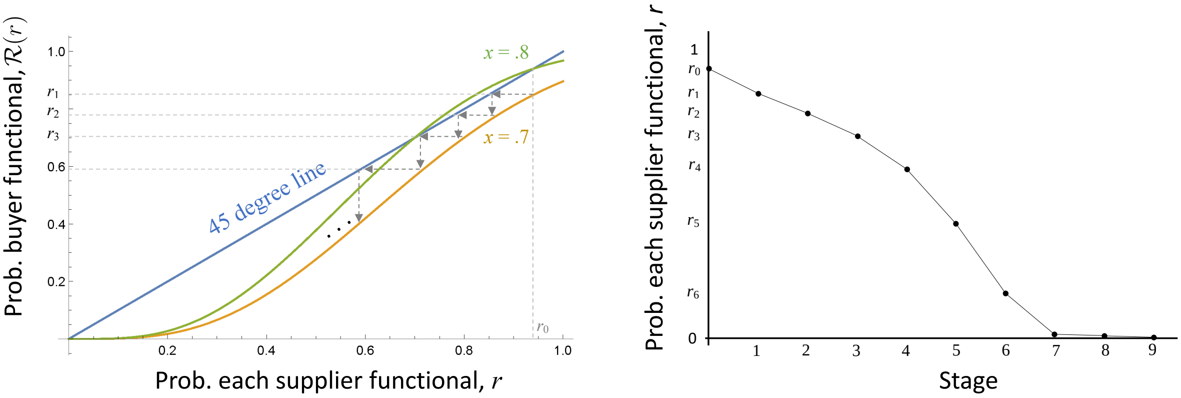
SA2. How production unravels when relationship strength is too low

Figure 3(b) in the main paper shows that when x drops below x_{crit}, the mass of firms that can consistently function falls discontinuously to $\rho(x) = 0$. While we will typically just work with the fixed point as the outcome of interest, the transition will not be instantaneous in practice. How then might the consequences of a shock to x actually play out?

In Figure 3, we work through a toy illustration to shed some light on the dynamics of collapse. Using the same parameters as our previous example, suppose relationship strength starts out at $x = 0.8$. The higher curve in panel (a) is $\mathcal{R}(\cdot; x)$ for this value of x. The reliability of the economy here is r_0, a fixed point of $\mathcal{R}$, which is the mass of functioning firms. Now suppose that a shock occurs, and all relationships become weaker, operating with the lower probability $x = 0.7$. The $\mathcal{R}$ curve now shifts, becoming the lower curve.

To consider the dynamics of how production responds, we must specify a few more details. We sketch one dynamic, and only for the purposes of this subsection. We interpret idiosyncratic link operation realizations as whether a given relationship works in a given period. Before the shift in x, a fraction r_0 of the firms are functional. Let $\widetilde{\mathcal{F}}(0)$ be the random set of functional firms at the time of the shock to x. Now x shifts to 0.7; we can view this as a certain fraction of formerly functional links failing, at random. Then firms begin reacting over a sequence of stages. Let us suppose that at stage s a firm can source its

[5]It is worth noting that this argument does not work for $m = 2$, in which case the numerator of $\frac{\partial h(n,2)}{\partial n}$ could be negative and thus $h(n,2)$ could be decreasing.



(A) (B)

FIGURE 3. The dynamics of unraveling (with the same parameters as in Figure 4 in the main paper, as discussed in Section SA2.

inputs if it has a functional link to a supplier who was functional in the last stage, $s-1$.[6] Let $\widetilde{\mathcal{F}}(s)$ be the set of these functional firms. By the same reasoning as in the previous subsection, we can see that the mass of $\widetilde{\mathcal{F}}(s)$, which we call r_s, is $\mathcal{R}(r_{s-1})$. Iterating the process leads to more and more firms being unable to produce as their suppliers fail to deliver essential inputs. After stage 1, the first set of firms that lost access to an essential input run out of stock and are no longer functional. This creates a new set of firms that cannot access an essential input, and these firms will be unable to produce at the end of the subsequent stage, and so on. The mass of each r_s can be described via the graphical procedure of Figure 3: take steps between the $\mathcal{R}$ curve and the 45-degree line.

This discussion helps make three related points. First, even though the disappearance of the positive fixed point—and thus the possibility of a positive mass of consistently functional firms—is sudden, the implications can play out slowly under natural dynamics.[7] The first few steps may look like a few firms being unable to produce, rather than a sudden and total collapse of output.

The second point is more subtle. Suppose that when the dynamic of the previous paragraph reaches r_2, the shock is reversed, x again becomes 0.8, and $\mathcal{R}$ again becomes the higher curve. Then, with some supply links reactivated, some of the firms that were made non-functional as the supply chain unravelled will become functional again, and this will allow more firms to become functional, and so on. Such dynamics could take the system back to the r_0 fixed point if sufficiently many firms remain functional at the time the shock is reversed. Thus, our theory predicts that *sufficiently persistent* shocks to relationship strength lead to eventual collapses of production, but, depending on the dynamics, the system may also be able to recover from sufficiently transient shocks.

The third point builds on the second. Suppose that a shock is anticipated and expected to be temporary. Then firms may take actions that slow the unravelling to reduce their amount of downtime. For example, they may build up stockpiles of essential inputs. If all firms behave in this way, the dynamics can be substantially slowed down and the possibility of recovery will improve.

Having illustrated some of the basic forces and timing involved in unraveling, we do not pursue here a more complete study of the dynamics of transient shocks, endogenous responses, etc.—an interesting subject in its own right. Instead, from now on we will focus on the size, $\rho(x)$, of the consistent functional set, which is the steady-state outcome under a relationship strength x.

[6]For example, firms might hold inventories that enable them to maintain production for a certain amount of time, even when unable to source an essential input. Even if firms engage in just-in-time production and do not maintain inventories of essential inputs, there can be a lag between shipments being sent and arriving.

[7]Indeed, in a more realistic dynamic, link realizations might be revised asynchronously, in continuous time, and firms would stop operating at a random time when they can no longer go without the supplier (e.g., when inventory runs out). Then the dynamics would play out "smoothly," characterized by differential equations rather than discrete iteration.

SA3. Interdependent supply networks and cascading failures

We now posit an interdependence among supply networks wherein each firm's profit depends on the *aggregate* level of output in the economy, in addition to the functionality of the suppliers with whom it has supply relationships. Formally, suppose, $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$, where $K_{\mathfrak{s}}$ is a strictly increasing function and Y is the integral across all sectors of equilibrium output:

$$Y = \int_{\mathcal{S}} \rho(x^*_{\mathfrak{s}}) d\Phi(\mathfrak{s}).$$

Here we denote by $x^*_{\mathfrak{s}}$ the unique positive equilibrium in sector $\mathfrak{s}$. The output in the sector is the reliability in that sector, $\rho(x^*_{\mathfrak{s}})$.

The interpretation of this is as follows: When a firm depends on a different sector, a specific supply relationship is not required, so the idiosyncratic failure of a given producer in the different sector does not matter—a substitute product can be readily purchased via the market. Indeed, it is precisely when substitute products are not readily available that the supply relationships we model are important. However, if some sectors experience a sudden drop in output, then other sectors suffer. They will not be able to purchase inputs, via the market, from these sectors in the same quantities or at the same prices. For example, if financial markets collapse, then the productivity of many real businesses that rely on these markets for credit are likely to see their effective productivity fall. In these situations, dependencies will result in changes to other sectors' profits even if purchases are made via the market. Our specification above takes interdependencies to be highly symmetric, so that only aggregate output matters, but in general these interdependencies would correspond to the structure of an intersectoral input-output matrix, and K would be a function of sector level outputs, indexed by the identity of the sourcing sector.

This natural interdependence can have very stark consequences. Consider an economy characterized by a distribution Ψ in which the subset of sectors with $m \geq 2$ has positive measure, and some of these have positive equilibria. Suppose that there is a small shock to $\underline{x}$. As already argued, this will directly cause a positive measure of sectors to fail. The failure of the fragile sectors will cause a reduction in aggregate output. Thus $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$ will decrease in other sectors *discontinuously.* This will take some other sectors out of the robust regime. Note that this occurs due to the other supply chains failing and not due to the shock itself. As these sectors are no longer robust, they topple too following an infinitesimal shock to $\underline{x}$. Continuing this logic, there will be a domino effect that propagates the initial shock. This domino effect could die out quickly, but need not. A full study of such domino effects is well beyond our scope, but the forces in the very simple sketch we have presented would carry over to more realistic heterogeneous interdependencies.

Fig. 4 shows[8] how an economy with 100 interdependent sectors responds to small shocks to $\underline{x}$. In this example, sectors differ only in their initial κ's. The technological complexity is set to $m = 5$ and the number of potential suppliers for each firm is set to $n = 3$. The cost function[9] for any firm if is $c(x_{if} - \underline{x}) = \frac{0.01}{(1-(x_{if}-\underline{x}))^2}$ while the gross profit function is $g(\rho(x)) = 5(1 - \rho(x))$. This setup yields values $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$ delimiting the region corresponding to critical (and therefore fragile) equilibria, as per Theorem 1.

In Fig. 4(a), the productivity shifter of a given sector is distributed uniformly, i.e. $\kappa_{\mathfrak{s}} \sim U(\underline{\kappa}, 25)$, so that many sectors have high enough productivity to be in a robust equilibrium while a small fraction have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors thus causes the failure of the fragile sectors (there are initially 12 of them). This then decreases the output Y across the whole economy, but only to a small extent (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is thus only enough to bring one robust sector into the fragile

[8]Note that in this example, the cascade dynamics is as follows: At step 1, firms in sectors with a κ in the fragile range fail due to an infinitesimal shock to $\underline{x}$. The initial economy-wide output Y_1 is then decreased to Y_2 and the κ's are updated using an updating function $K(Y)$ increasing in Y. Only then, are the firms in the surviving sectors allowed re-adjust x_{if}. At step 2, infinitesimal shocks hit again and the firms newly found in the fragile regime fail. This process goes on at each step until no further firm fails, at which point the cascade of failures stops.

[9]For simplicity, we set $\underline{x} = 0$. An infinitesimal shock to $\underline{x}$ has the effect of causing the firms of sectors in the fragile regime to fail, but does not affect the value of $\underline{x}$, which remains at 0.

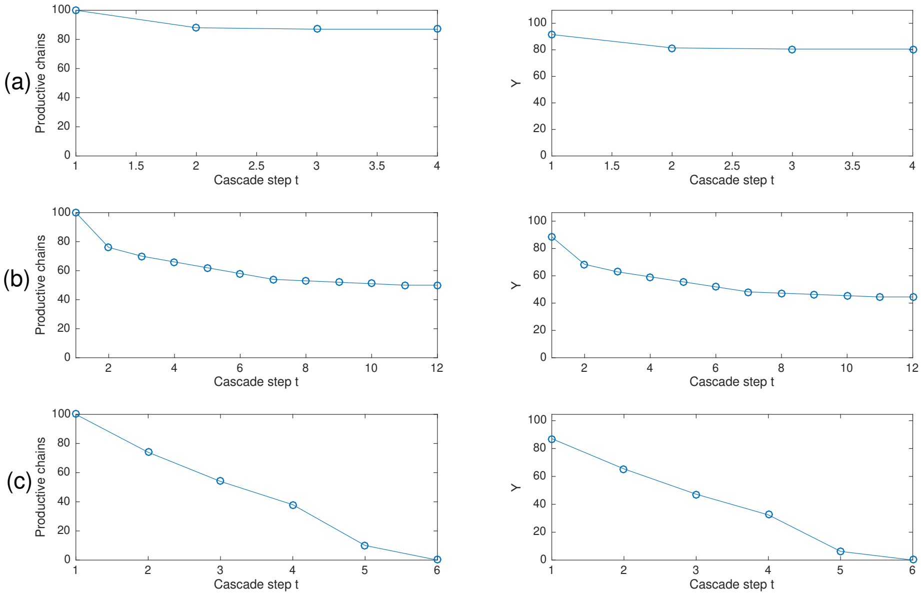


FIGURE 4. Number of sectors that remain productive (left) and economy-wide output Y (right) for each step of a cascade of failures among 100 interdependent sectors. For all sectors: $n = 3$, $m = 5$, $c(x_{if} - \underline{x}) = \frac{0.01}{(1-(x_{if}-\underline{x}))^2}$, $g(\rho(x)) = 5(1 - \rho(x))$. This yields $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$. In row (a), 100 sectors have $\kappa_{\mathbf{s}}$ initially distributed according to $U(\underline{\kappa}, 25)$; In row (b), 100 sectors have $\kappa_{\mathbf{s}}$ initially distributed according to $U(\underline{\kappa}, 13)$; In row (c), 100 sectors have $\kappa_{\mathbf{s}}$ initially distributed according to $U(\underline{\kappa}, 10)$.

regime and thus to cause it to fail as well upon a small shock. In the end, a total of only 13 sectors have failed.

In contrast, Fig. 4(b) shows an economy where $\kappa_{\mathbf{s}} \sim U(\underline{\kappa}, 13)$, so that more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors causes the failure of the fragile sectors (now initially 24). These have a larger effect on decreasing the output Y across the whole economy (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is now enough to bring many of them into the fragile regime and to cause them to fail upon an infinitesimal shock to $\underline{x}$. This initiates a cascade of sector failures, ultimately resulting in 50 sectors ceasing production.

Fig. 4(c) shows an economy where $\kappa_{\mathbf{s}} \sim U(\underline{\kappa}, 10)$, so that even more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the $\underline{x}$ of all sectors causes the failure of the fragile sectors (now initially 26) and this initiates a cascade of sector failures which ultimately brings down all 100 sectors of the economy.

The discontinuous drops in output caused by fragility, combined with the simple macroeconomic interdependence that we have outlined, come together to form an amplification channel reminiscent, e.g., of Elliott, Golub, and Jackson (2014) and Baqaee (2018). Thus, the implications of those studies apply here: both the cautions regarding the potential severity of knock-on effects, as well as the importance of preventing first failures before they can cascade.

SA4. Production trees with limited depth

The main results of the paper are stated for supply chain depth d sufficiently large. In this section, we numerically explore the shape of the reliability function $\tilde{\rho}(x, d)$ at realistic values of d. We do so while allowing for some systematic heterogeneity (for example, more upstream tiers being simpler).

To motivate the literal modeling of assembly in finitely many tiers, we return to the example of an Airbus A380. This product has 4 million parts. The final assembly in Toulouse, France, consists of six large components coming from five different factories across Europe: three fuselage sections, two wings, and the horizontal tailplane. Each of these factories gets parts from about 1500 companies located in 30 countries[10]. Each of those companies itself has multiple suppliers, as well as contracts to supply and maintain specialized factory equipment, etc.

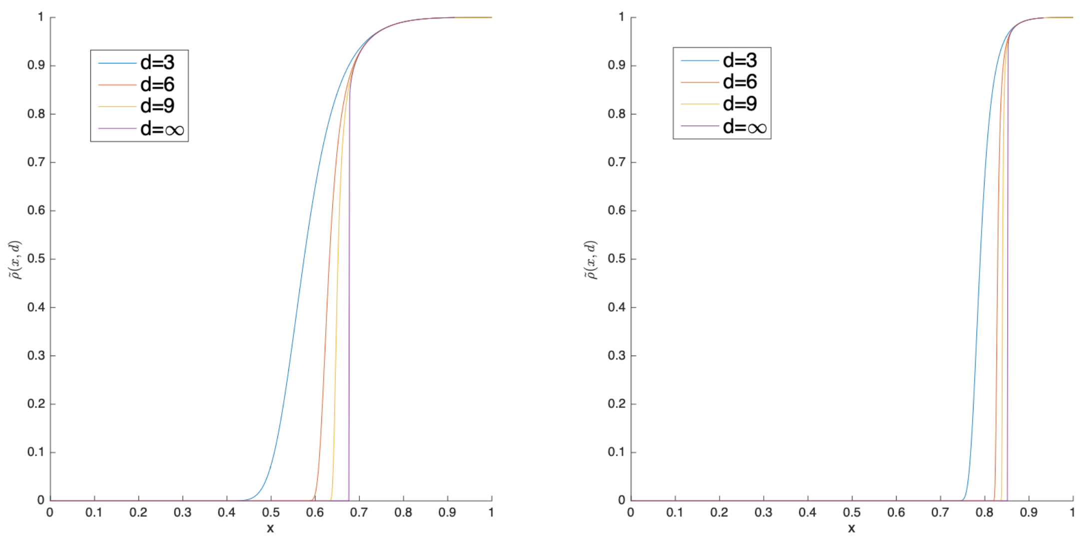
As in the main paper consider a depth-d supply tree, but let each firm in tier $t \in \{0, 1, ..., d\}$ require m_t kinds of inputs and have n_t potential suppliers of each input. Here $t = d$ is the most downstream tier and $t = 0$ is the most upstream tier. The nodes at tier $t = 0$ are functional for sure. As before, we denote by $\widetilde{\rho}(x, d)$ the probability of successful production at the most downstream node of a depth-d tree with these properties. This is defined as

$$\widetilde{\rho}(x, d) = (1 - (1 - \widetilde{\rho}(x, d-1)x)^{n_d})^{m_d}$$

with $\widetilde{\rho}(x, 0) = 1$, since the bottom-tier nodes do not need to obtain inputs.

We see that the expression is recursive and, if unraveled explicitly, would be unwieldy after a number of tiers. However, we know from Definition 3 and Proposition 5 (both in the paper) that when $m_t = m$ and $n_t = n$ for all t, then for any $x \in [0, 1]$, as d goes to infinity, $\widetilde{\rho}(x, d)$ converges to the correspondence $\rho(x)$.

We start with some examples where m_t and n_t are the same throughout the tree. Figure 5 illustrates the successful production probability $\widetilde{\rho}(x, d)$ for different finite depths d and how quickly it converges to the correspondence $\rho(x)$.



(A) (B)

FIGURE 5. Successful production probability $\widetilde{\rho}(x, d)$ for different finite numbers of tiers d. In panel (A), $m = 5$ and $n = 4$. In panel (B), $m = 40$ and $n = 4$.

In panel (A), we see that $\widetilde{\rho}(x, d)$ exhibits a sharp transition for a depth as small as $d = 3$. The red curve ($d = 6$) shows that when the investment level x drops from 0.66 to 0.61, or about 7 percent, production probability $\widetilde{\rho}(x, 6)$ drops from 0.8 to 0.1. (Thus $\widetilde{\rho}(x, 6)$ achieves a slope of at least 14.) In panel (B), we see that increasing product complexity (by increasing m to 40) causes $\widetilde{\rho}(x, d)$ to lie quite close to $\rho(x)$. This illustrates how complementarities between inputs play a key role in driving this sharp transition in the probability of successful production. Note that $m = 40$ is not an exaggerated number in reality. In the Airbus example described earlier, many components would exhibit such a level of complexity.

However, a complexity number like $m = 40$ will not occur everywhere throughout the supply network. Indeed, and more generally, one might ask whether the regularity in the production tree is responsible for the sharpness of the transition. To investigate this possibility, in Figure 6 we plot $\widetilde{\rho}(x, d)$ for a supply

[10]Source: "FOCUS: The extraordinary A380 supply chain". *Logistics Middle East*. Retrieved on 28 may, 2019 from `https://www.logisticsmiddleeast.com/article-13803-focus-the-extraordinary-a380-supply-chain`

tree with irregular complexity, where different tiers may have different values of m_t. Here we construct 4 trees whose complexity increases with d. The first trees has $d = 3$ and $m_1 = 2$, $m_2 = 6$ and $m_3 = 10$. The second tree has $d = 6$, $m_t = 2$ for $t = 1, 2$, $m_t = 6$, for $t = 3, 4$ and $m_t = 10$, for $t = 5, 6$. The third tree has $d = 9$, $m_t = 2$ for $t = 1, 2, 3$, $m_t = 6$, for $t = 4, 5, 6$ and $m_t = 10$, for $t = 7, 8, 9$. Finally, the fourth tree has large depth (here $d = 999$), $m_t = 2$ for t below the first tercile, $m_t = 6$ for t between the first and the second terciles and $m_t = 10$ for t above the second tercile. We see that trees of moderate depth once again exhibit a sharp transition in their probability of successful production. This feature is thus not at all dependent upon the regularity of the trees.

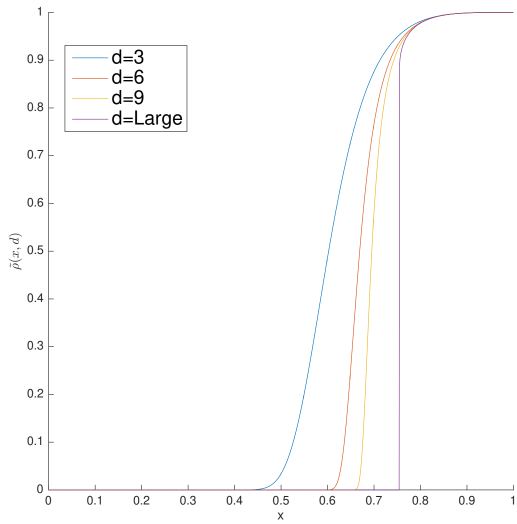


FIGURE 6. Successful production probability $\tilde{\rho}(x, d)$ for different finite numbers of tiers d, but where different tiers may have different complexity m.

SA5. HETEROGENEITY

In this section, we first present a result showing that the sharp transition in production probability arises in the heterogeneous setting (Proposition SA1).

We then present two examples exhibiting the fragility and weakest link properties (Subsection SA5.3). We then explain how we numerically solved the examples (Subsection SA5.4) and report some additional information about them.

SA5.1. Generalization of the sharp transition in the heterogeneous case.

The next result shows that the basic physical implications of supply chain complexity are robust to heterogeneity. In the general environment described in Section 6.2.6 in the paper, there is an analogue of Proposition 1.

To formalize this we introduce just for simplicity a single parameter ξ reflecting institutional quality. We posit that $x_{if,j} = \mathrm{x}_{\mathrm{ij}}(\xi)$, where x_{ij} are strictly increasing, differentiable, surjective functions $[0, 1] \to [0, 1]$.

PROPOSITION **SA1.** Suppose that for all products i, the complexity m_i is at least 2. Moreover, suppose whenever $j \in I(i)$, the number n_{ij} of potential suppliers for each firm is at least 1. For any product i, the measure of the set of functional firms $\overline{\mathcal{F}}_i$, denoted by $\rho_i(\xi)$, is a nondecreasing function with the following properties.

(1) There is a number ξ_{crit} and a vector $\boldsymbol{r}_{\mathrm{crit}} > 0$ such that $\boldsymbol{\rho}$ has a discontinuity at ξ_{crit}, where it jumps from 0 to $\boldsymbol{r}_{\mathrm{crit}}$ and is strictly increasing in each component after that.
(2) If $n_{ij} = 1$ for all i and j, we have that $\xi_{\mathrm{crit}} = 1$; otherwise $\xi_{\mathrm{crit}} < 1$.
(3) If $\xi_{\mathrm{crit}} < 1$, then as ξ approaches ξ_{crit} from above, the derivative $\rho_i'(x)$ tends to ∞ in some component.

The idea of this result is simple, and generalizes the graphical intuition of Figure 4 in Section 3.1.1 of the paper. For any $\boldsymbol{r} \in [0,1]^{|\mathcal{I}|}$, define $\mathcal{R}^{\xi}(\boldsymbol{r})$ to be the probability, under the parameter ξ, that a producer of product i is functional given that the reliability vector for producers of other products is given by $\boldsymbol{r}$. This can be written explicitly:

$$[\mathcal{R}^{\xi}(\boldsymbol{r})]_i = \prod_{j \in I(i)} \left[1 - (1 - r_j \mathrm{x}_{ij}(\xi))^{n_{ij}}\right].$$

By Tarski's theorem, there is a pointwise largest fixed point of this function and by the same reasoning as in Section 3.1.1 of the paper, it corresponds to the reliabilities in the deep-network limit.

Near 0, the map $\mathcal{R}^{\xi} : [0,1]^n \to [0,1]^n$ is bounded above by a quadratic function (as a consequence of $m_i \geq 2$ for all i). Therefore it cannot have any fixed points near 0. Thus, analogously to Figure 4 in the main paper, fixed points disappear abruptly as ξ is reduced past a critical value ξ_{crit}.

Proof of Proposition SA1. For any $\boldsymbol{r} \in [0,1]^{|\mathcal{I}|}$, define $\mathcal{R}^{\xi}(\boldsymbol{r})$ to be the probability, under the parameter ξ, that a producer of product i is functional given that the reliability vector for all products is given by $\boldsymbol{r}$. This can be written explicitly:

$$[\mathcal{R}^{\xi}(\boldsymbol{r})]_i = \prod_{j \in I(i)} \left[1 - (1 - r_j \mathrm{x}_{\mathrm{ij}}(\xi))^{n_{ij}}\right].$$

Let $\boldsymbol{\rho}(\xi)$ be the elementwise largest fixed point of $\mathcal{R}^{\xi}$, which exists and corresponds to the mass of functional firms by the same argument as in Lemma 2 in the main paper.

(1) It is clear that $\boldsymbol{\rho}(1) = \mathbf{1}$.

(2) Next, there is an $\epsilon > 0$ such that if $\|\boldsymbol{r}\| < \epsilon$, then for all ξ, the function $\mathcal{R}^{\xi}(\boldsymbol{r}) < \boldsymbol{r}$ elementwise. So there are no fixed points near $\mathbf{0}$.

(3) For small enough ξ, the function $\mathcal{R}^{\xi}$ is uniformly small, so $\boldsymbol{\rho}(\xi) = \mathbf{0}$.

These facts together imply that $\boldsymbol{\rho}$ has a discontinuity where it jumps up from $\mathbf{0}$. Let ξ_{crit} be the infimum of the ξ where $\boldsymbol{\rho}(\xi) \neq 0$.

Define

$$\Gamma(\mathcal{R}^{\xi}(\boldsymbol{r})) = \{(\boldsymbol{r}, \mathcal{R}^{\xi}(\boldsymbol{r})) : \boldsymbol{r} \in [0,1]^{|\mathcal{I}|}\}$$

to be the graph of the function. What we have just said corresponds to the fact that this graph intersects the diagonal when $\xi = \xi_{\text{crit}}$, but not for values $\xi < \xi_{\text{crit}}$. Suppose now, toward a contradiction, that the derivative $\boldsymbol{\delta}(\xi)$ of $\boldsymbol{\rho}(\xi)$ is bounded in every coordinate as $\xi \downarrow \xi_{\text{crit}}$. Then, passing to a convergent subsequence and using the smoothness of $\mathcal{R}$, we find that the derivative $\boldsymbol{\delta}(\xi_{\text{crit}})$ of $\boldsymbol{\rho}$ is well-defined at $\xi = \xi_{\text{crit}}$. But that contradicts our earlier deduction that $\boldsymbol{\rho}$ is discontinuous at ξ_{crit}. □

SA5.2. **Proof of Proposition 4 (Weakest links).** Here we prove Proposition 4 in the main text.

Consider an equilibrium x^* with reliabilities r^*.

Part (i): Let $\mathcal{P}$ be a directed path of length T from node T to node 1 and denote product i by 1. Since product 1 is critical, then following a shock $\epsilon > 0$ to γ_{ij}, $r_1^{*'} = 0$, where the 'prime' notation denotes the equilibrium quantity after the shock.

For any product $t+1$ that sources input t and such that $r_t^{*'} = 0$, we have that

$$\begin{aligned} r_{t+1}^{*'} &= \prod_{l \in I(t+1)} (1 - (1 - x_{t+1,l}^* r_l^{*'})^{n_{t+1,l}}) \\ &= 0 \end{aligned}$$

since $t \in I(t+1)$.

Since $r_1^{*'} = 0$, it then follows by induction that the production of all products $t \in \mathcal{P}$ will fail.

Part (ii): Suppose production of some product $i \in \mathcal{I}^{SC}$ is critical and consider another product $k \in \mathcal{I}^{SC}$ that is an input for the production of product i (that is, $k \in I(i)$). As an investment equilibrium x^* is being played, the strength x_{kj} of a link from a producer of product k to a supplier of input $j \in I(k)$ must satisfy the following condition

$$MB_{kj} = \kappa g(r_k) \prod_{l \in I(k), l \neq j} (1 - (1 - x_{kl} r_l)^{n_{kl}}) n_{kj} (1 - x_{kj} r_j)^{n_{kj}-1} r_j = \gamma_{kj} \widetilde{c}'(x_{kj} - \underline{x}_{kj}) = MC_{kj}$$

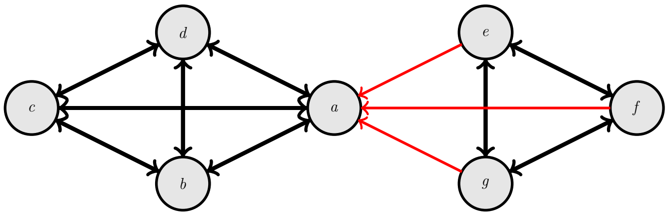


FIGURE 7. Supply dependencies: Black bold arrows represent reciprocated supply dependencies in which both products require inputs from each other. A red arrow from one product to another means that the product at the origin of the arrow uses as an input the product at the end of the arrow (e.g. product e requires product a as an input, but not the other way around). Product a also depends on itself, but the corresponding self-link is not shown.

Rearranging this equation yields

$$\frac{\kappa g(r_k^*)}{\gamma_{kj}} \prod_{l \in I(k), l \neq j} (1-(1-x_{kl}^* r_l^*)^{n_{kl}}) n_{kj} r_j^* = \frac{\tilde{c}'(x_{kj}^* - \underline{x}_{kj})}{(1-x_{kj}^* r_j^*)^{n_{kj}-1}}.$$

The right hand side is strictly increasing in x_{kj}^*, while the left hand side is constant.

Consider now a shock $\epsilon > 0$ that changes the value of γ_{kj} to $\gamma'_{kj} = \gamma_{kj} + \epsilon$. This strictly reduces $\kappa g(r_k^*)/\gamma_{kj}$, and hence the new equilibrium investment level satisfies $x_{kj}^{*'} < x_{kj}^*$. This in turn implies that $r_k^{*'} < r_k^*$, and so

$$r_i^{*'} = \prod_{l \in I(i)} (1-(1-x_{i,l}^* r_l^{*'})^{n_{i,l}}) \tag{SA-7}$$

$$< \prod_{l \in I(i)} (1-(1-x_{i,l}^* r_l^*)^{n_{i,l}}) \tag{SA-8}$$

$$= r_i^* \tag{SA-9}$$

since $k \in I(i)$. Thus $r_i^{*'} = 0$ and the production of product i fails.

Now since both products k and i are part of a strongly connected component, there is also a directed path from k to i. From part (i), it follows that every product t on such a directed path (i.e. every product that uses input i either directly or indirectly through intermediate products) will also have $r_t^{*'} = 0$. This is true namely for product k and thus $r_k^{*'} = 0$. We therefore conclude that, following a small decrease in its sourcing effort from the initial x_k^*, production of product k fails. Product k was thus necessarily critical and we must have had x_k^* be critical.

Proceeding similarly for any other product $e \in I(k)$ such that $e \in \mathcal{I}^{SC}$, we get that e is also critical. By induction it follows that all products in a strongly connected component $\mathcal{I}^{SC}$ of the product interdependencies graph are critical when one of them is. We conclude that either production of all the products in $\mathcal{I}^{SC}$ are critical or else production of all the products in $\mathcal{I}^{SC}$ are noncritical. □

SA5.3. **Two examples with heterogeneities.**

EXAMPLE **SA1.** There are seven products. Only product a is used as an input into its own production. Products a to d all use inputs from each other, products e to g all use inputs from each other, and products e to g also require product a as an input. Figure 7 shows the input dependencies between these products. We let products a and b have three potential suppliers for each of their required inputs, while products c to g have only two potential suppliers for each of their required inputs. We let the profitability of the seven products differ systematically. Specifically, we let $G_i(r_i) = \alpha_i(1 - r_i)$ and set

$$\alpha = [2857.4, 2456.1, 53.5, 43.9, 5.8, 5.5, 5.1].$$

We let the cost of a producer of product i from investing in supplier relationships with producers of product j be $c_{ij}(x_{ij} - \underline{x}_{ij}) = \frac{1}{2}\gamma_{ij} x_{ij}^2$ (where, for simplicity, we have set $\underline{x}_{ij} = 0$), and we set, for now, $\gamma_{ij} = 1$ for all product pairs ij.

The equilibrium investments that firms must make are pinned down by equating the marginal costs and benefits of each investment for each firm. For this configuration, production of products e, f and g

is critical, while production is non-critical for products a, b, c and d. We report the equilibrium levels of investment, along with gross and net profits, in Section SA5.4.

We then model a small unanticipated shock to the cost of firms producing product i investing in their relationship strength with suppliers of product j, by increasing γ_{ij} from 1 to $1+\epsilon$. After the shock to γ_{ij}, producers of product i choose to exert less effort sourcing input j. For γ_{ij} such that $i = a, b, c$ or d, the impact of this is minor. The probability of successful production for those products (r_i) only drops continuously and for a small shock the change will be small. Nevertheless, this is still sufficient for the output of the firms producing products e, f and g to collapse to 0 since they are 'critical' and they source (directly or indirectly) products a, b, c and d.

Similarly, if the shock occurs to the sourcing efforts of firms producing products e, f or g, output of these products will collapse to 0 since they are 'critical'. On the other hand, output of products a, b, c and d will not be affected since these producers do not require inputs e, f or g.

Example **SA2.** We adjust the configuration of the previous example by letting the vector of product profitabilities be

$$\alpha = [32.46, 45.37, 8.52, 9.24, 21.78, 24.62, 28.00].$$

Everything else, remains the same as before.

Given these parameters we numerically solve for an equilibrium. For these parameter values production of products a, b, c and d is now critical, while production of products e, f and g is not.

Consider now a shock to γ_{ij} for $i \in \{a, b, c, d\}$. Output of the shocked product i then collapses to 0, and thus so will the output of the firms producing products $\{a, b, c, d\}$. Output of products e, f and g will then also collapse to 0 since those producers all source (directly or indirectly) inputs a, b, c and d.

Consider now a shock to γ_{ij} for $i \in \{e, f, g\}$. Output of the shocked product will adjust to accommodate the shock and the probability of successful production for the affected product will fall continuously. For a small shock, this decrease in output will be small. Products a, b, c and d will be unaffected even though they are critical, since their firms do not source products e, f and g.

SA5.4. **Solving the examples with heterogeneities numerically.** To compute the equilibria in Examples SA1 and SA2, we proceed as follows.

As before we consider symmetric equilibria, in the sense that all producers of product i invest the same amount sourcing a given input j. As we are interested in these symmetric investment choices, we dispense with the subscript f when we consider investments. In such an equilibrium the profit of a producer of product i is given by

$$\Pi_i = G_i(r_i)r_i - \frac{1}{2}\sum_{j\in I(i)} \gamma_{ij}x_{ij}^2, \tag{SA-10}$$

where

$$r_i = \prod_{j\in I(i)} (1-(1-x_{ij}r_j)^{n_{ij}}),$$

$I(i)$ is the neighborhood of i on the product dependency graph, with $|I(i)| = m_i$ (the complexity of production for product i), and n_{ij} is the number of potential suppliers a producer of product i has for input j (i.e. the potential level of multisourcing by producers of product i for input j). Note also that in equation (SA-10), the second term represents the effort cost function and we have set $\underline{x}_{ij} = 0$ for simplicity.

The marginal benefit a producer of product i receives from investing in its relationships with suppliers of input j is

$$MB_{ij} = G_i(r_i) \prod_{l\in I(i), l\neq j} (1-(1-x_{il}r_l)^{n_{il}})n_{ij}(1-x_{ij}r_j)^{n_{ij}-1}r_j. \tag{SA-11}$$

Letting $\gamma_{ij} = 1$ (as in the examples), the marginal cost for a producer of product i investing in a relationship with a supplier of input j is

$$MC_{ij} = x_{ij}. \tag{SA-12}$$

We look for $|\mathcal{I}|\times|\mathcal{I}|$ matrix X (i.e., the matrix containing the investment profiles for all products), with entries x_{ij} satisfying $MB_{ij} = MC_{ij}$. The value of x_{i1} that equates the marginal benefits and marginal costs for a producer of product i's investment into sourcing product 1 is increasing in $G_i(r_i) = \alpha_i(1-r_i)$, which a producer of product i takes a given. As we still have the freedom to choose α_i we can select an arbitrary $x_{i1} \in (0,1)$. However, doing so pins down the value of x_{ij} for all $j \neq 1$. Specifically, we must have

$$\frac{MB_{ij}}{MB_{i1}} = \frac{MC_{ij}}{MC_{i1}}, \quad \forall i, j$$

which can be expressed as

$$\frac{G(r_i)\prod_{l\in I(i),l\neq j}(1-(1-x_{il}r_l)^{n_{il}})n_{ij}(1-x_{ij}r_j)^{n_{ij}-1}r_j}{G(r_i)\prod_{l\in I(i),l\neq 1}(1-(1-x_{il}r_l)^{n_{il}})n_{i1}(1-x_{i1}r_1)^{n_{i1}-1}r_1} = \frac{x_{ij}}{x_{i1}},$$

and reduces to

$$\frac{(1-x_{ij}r_j)^{n_{ij}-1}}{(1-(1-x_{ij}r_j)^{n_{ij}})}\frac{n_{ij}}{x_{ij}} = \frac{n_{i1}}{x_{i1}}\frac{r_1}{r_j}\frac{(1-x_{i1}r_1)^{n_{i1}-1}}{(1-(1-x_{i1}r_1)^{n_{i1}})}. \tag{SA-13}$$

The left-hand side is decreasing in x_{ij} while the right-hand side is given, so there can be only one solution x_{ij} satisfying the above.

We initialize x_{i1} for all i to values just smaller than 1. We then decrease these values incrementally by a small amount. After each reduction we calculate the X matrix using the above procedure and calculate the probability of successful production r_i for each product. We continue until the probability of successful production decreases to 0 for one of the products i. This gives us values of X (i.e., the investment profiles for all products) such that at least one product is in the critical regime.

The values of $G_i(r_i)$ are then set so that $MB_{i1} = MC_{i1}$ using equations (SA-11) and (SA-12). This ensures that all firms are choosing profit maximizing investments that result in at least one product being fragile. Recall that $G_i(r_i) = \alpha_i(1-r_i)$, and so depends on α_i. Thus, for a given value of $G_i(r_i)$ we set $\alpha_i = \frac{G_i(r_i)}{(1-r_i)}$.

Using this procedure there turn out to be essentially two types of equilibria with fragile firms. Either a firm in the set $\{a,b,c,d\}$ becomes fragile first, in which case all firms in this set become fragile simultaneously, or else a firm in the set $\{e,f,g\}$ becomes fragile first in which case all firms in this set simultaneously become fragile. This follows directly from Proposition 4. When a firm in the set $\{a,b,c,d\}$ becomes fragile first, a shock to any one of $\{a,b,c,d\}$ that reduces the reliability of sourcing an input (either directly, or indirectly by reducing incentives to invest in reliability) is sufficient for the probability of successful production of *all* firms to fall to 0. When a firm in the set $\{e,f,g\}$ becomes fragile first, a similar shock to any one of these firms is sufficient for the probability of successful production of these firms, but not firms $\{a,b,c,d\}$, to fall to 0. The parameters selected in Examples 1 and 2 are chosen to illustrate these two possible cases.

Example SA1—additional information The equilibrium relationship strengths are reported in the matrix X below, where an entry x_{ij} represents the strength chosen by a producer of product i in a relationship sourcing input j.

By pinning down the first column of X with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.8873 & 0.8872 & 0.9315 & 0.9385 & 0 & 0 & 0 \\ 0.8773 & 0 & 0.9204 & 0.9272 & 0 & 0 & 0 \\ 0.8673 & 0.8672 & 0 & 0.9084 & 0 & 0 & 0 \\ 0.8573 & 0.8572 & 0.8915 & 0 & 0 & 0 & 0 \\ 0.7573 & 0 & 0 & 0 & 0 & 0.9726 & 0.9783 \\ 0.7473 & 0 & 0 & 0 & 0.9464 & 0 & 0.9572 \\ 0.7373 & 0 & 0 & 0 & 0.9265 & 0.9317 & 0 \end{bmatrix}.$$

Such an X corresponds to the following product reliabilities:

$$r = [0.9926, 0.9928, 0.9387, 0.9307, 0.5384, 0.5262, 0.5145].$$

Also, for such values of x_{ij}, products a, b, c, d are non critical, while products e, f, g are critical.

We can obtain $G = [21.0836, 17.7538, 3.2818, 3.0451, 2.6780, 2.5859, 2.4990]$.

Recall that we let $G_i(r_i) = \alpha_i(1 - r_i)$. Computing $\alpha_i = \frac{G_i(r_i)}{1-r_i}$, we obtain

$$\alpha = [2857.4, 2456.1, 53.5, 43.9, 5.8, 5.5, 5.1].$$

The net profits of producers of each product are

$$\Pi = [19.2666, 16.3872, 1.9159, 1.7016, 0.2036, 0.1756, 0.1506].$$

Example SA2—additional information The equilibrium investment levels are reported in the matrix X below, where an entry x_{ij} represents the investment made by a producer of product i towards sourcing input j.

By pinning down the first column of X with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.7965 & 0.7792 & 0.8735 & 0.8663 & 0 & 0 & 0 \\ 0.8065 & 0 & 0.8859 & 0.8785 & 0 & 0 & 0 \\ 0.8165 & 0.8029 & 0 & 0.8681 & 0 & 0 & 0 \\ 0.8265 & 0.8124 & 0.8855 & 0 & 0 & 0 & 0 \\ 0.8965 & 0 & 0 & 0 & 0 & 0.8947 & 0.8894 \\ 0.9065 & 0 & 0 & 0 & 0.9103 & 0 & 0.8992 \\ 0.9165 & 0 & 0 & 0 & 0.9204 & 0.9146 & 0 \end{bmatrix}.$$

Such an X corresponds to the following product reliabilities:

$$r = [0.8837, 0.9132, 0.7653, 0.7756, 0.8778, 0.8865, 0.8951].$$

Also, for such values of x_{ij}, the production of products a, b, c, d is critical, while the production of products e, f, g is now non critical.

We can obtain $G = [3.7758, 3.9399, 1.9995, 2.0736, 2.6608, 2.7929, 2.9372]$.

Recall that we let $G_i(r_i) = \alpha_i(1 - r_i)$. Computing $\alpha_i = \frac{G_i(r_i)}{1-r_i}$, we obtain

$$\alpha = [32.46, 45.37, 8.52, 9.24, 21.78, 24.62, 28.00].$$

The net profits of producers of each product are

$$\Pi = [1.9590, 2.4942, 0.4978, 0.5446, 1.1381, 1.2466, 1.3673].$$

SA6. Interpretation of investment

SA6.1. **Effort on both the extensive and intensive margins.** This section supports the claims made in Section 6.2.1 of the paper that our model is easily extended to allow firms to make separate multi-sourcing effort choices on the intensive margin (quality of relationships) and the extensive margin (finding potential suppliers).

Suppose a firm if chooses efforts $\widehat{e}_{if} \geq 0$ on the extensive margin and effort $\widetilde{e}_{if} \geq 0$ on the intensive margin, and suppose that $x_{if} = h(\widehat{e}_{if}, \widetilde{e}_{if})$. Let the cost of investment be a function of $\widehat{e}_{if} + \widetilde{e}_{if}$ instead of y_{if}. This firm problem can be broken down into choosing an overall effort level $e_{if} = \widehat{e}_{if} + \widetilde{e}_{if}$ and then a share of this effort level allocated to the intensive margin, with the remaining share allocated to the extensive margin. Fixing an effort level e, a firm will choose $\widehat{e}_{if} \in [0, e]$, with $\widetilde{e}_{if} = e - \widehat{e}_{if}$, to maximize x_{if}. Let $\widehat{e}^*_{if}(e)$ and $\widetilde{e}^*_{if}(e) = e - \widehat{e}^*_{if}(e)$ denote the allocation of effort across the intensive and extensive margins that maximizes x_{if} given overall effort e. Given these choices, define $h^*(e) := h(\widehat{e}^*_{if}(e), \widetilde{e}^*_{if}(e))$. As h^* is strictly increasing in e, choosing e is then equivalent to choosing x_{if} directly, with a cost of effort equal to $c(h^{*-1}(e))$. Thus, as long as the cost function $\widehat{c}(e) := c(h^{*-1}(e))$ continues to satisfy our maintained assumptions on c, everything goes through unaffected.

SA6.2. **A richer extensive margin model.** In the previous subsection we gave an extensive margin search effort interpretation of x_{if}. In some ways this interpretation was restrictive. Specifically, it required there to be exactly n suppliers capable for supplying the input and that each such supplier be found independently with probability x_{if}. This alternative interpretation is a minimal departure from the intensive margin interpretation, which is why we gave it. However, it is also possible, through a change of variables, to see that our model encompasses a more general and standard search interpretation.

Fixing the environment a firms faces, specifically the probability other firms successfully produce $r > 0$ and a parameter n that will index the ease of search, suppose we let each firm if choose directly the probability that, through search, it finds an input of given type. When $r = 0$ we suppose that all search is futile and that firms necessarily choose $\widehat{x}_{if} = 0$. Denote the probability firm if finds a supplier of a given input type by $\widehat{x}_{if}$. Conditional on finding an input, we let it be successfully sourced with probability 1 so all frictions occur through the search process. Implicitly, obtaining a probability $\widehat{x}_{if}$ requires search effort, and we suppose that cost of achieving probability $\widehat{x}_{if}$ is $\widehat{c}(\widehat{x})$, where $\widehat{c}$ is a strictly increasing function with $\widehat{c}(0) = 0$.

We suppose firms choose $\widehat{x}_{if}$ taking the environment as given. In particular, firms take as given the probability that suppliers of the inputs they require successfully produce. When many potential suppliers of an input produce successfully we let it be relatively easy to find one, and if none of these suppliers produce successfully then it is impossible to find one. In addition, the parameter n shifts how easy it is to find a supplier.

Given this set up we can let the probability of finding a supplier have the functional form $\widehat{x} := 1 - (1 - x_{if} r)^n$, and the cost of achieving this probability be given by $\widehat{c}(\widehat{x}) := c\left(\frac{1-(1-\widehat{x})^{1/n}}{r}\right)$. Although these functional form assumptions might seem restrictive, we still have freedom to use any function c satisfying our maintained assumptions. This degree of freedom is enough for the model to be quite general as all that matters is the size of the benefits of search effort relative to its cost, and not the absolute magnitudes. Further, these functional form assumptions satisfy all the desiderata we set out above. As $1 - (1 - x_{if} r)^n$ is the key probability throughout our analysis, all our results then go through with this interpretation.

References

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