ONLINE APPENDIX:
A NETWORK APPROACH TO PUBLIC GOODS

Throughout the online appendix, we refer often to sections, results, and equations in the main text and its appendix using the numbering established there (e.g., Section 2.2, Appendix A, equation (4)). The numbers of sections, results, and equations in this online appendix are all prefixed by OA to distinguish them, and we always use this prefix in referring to them.

OA1. Multiple Actions

This section extends our environment to permit each agent to take actions in multiple dimensions, and then proves analogues of our main results. We focus on what we consider to be the essence of our analysis—namely the equivalence of certain eigenvalue properties, and certain matrix equations, to efficient and Lindahl outcomes. Other important matters—existence of efficient and Lindahl points, as well as their strategic microfoundations—are not treated here, but we believe that the techniques introduced in the main text would establish analogous results.

OA1.1. **Environment.** We adjust the environment only by permitting players to take multi-dimensional actions $\mathbf{a}_i \in \mathbb{R}^k_+$, with entry d of player i's action vector being denoted by a_i^d. Each player then has a utility function $u_i : \mathbb{R}^{nk}_+ \to \mathbb{R}$. When we need to think of $\mathbf{a}$ as a vector—i.e., when we need an explicit order for its coordinates—we will use the following one. First we list all actions on the first dimension, then all actions on the second dimension, etc.:

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^{[1]}_{\cdot} \\ \mathbf{a}^{[2]}_{\cdot} \\ \vdots \\ \mathbf{a}^{[k]}_{\cdot} \end{bmatrix}.$$

For each $d \in \{1, 2, \ldots, k\}$, we construct the n-by-n Jacobian $\mathbf{J}^{[d]}(\mathbf{a})$ by setting $J^{[d]}_{ij}(\mathbf{a}) = \partial u_i(\mathbf{a})/\partial a_j^d$. We define the benefits matrix:

$$B^{[d]}_{ij}(\mathbf{a};\mathbf{u}) = \begin{cases} \frac{J^{[d]}_{ij}(\mathbf{a};\mathbf{u})}{-J^{[d]}_{ii}(\mathbf{a};\mathbf{u})} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

The following assumptions are made on these new primitives. First, utility functions are concave and continuously differentiable. Second, all actions are costly.[1] Third, there are weakly positive externalities from all actions.[2] Fourth, benefit flows are connected, so that each matrix $\mathbf{B}^{[d]}(\mathbf{a})$ is irreducible, for all $\mathbf{a}$. These assumptions are very similar to those we required in the one-dimensional case.

Date: May 16, 2014.

[1] $\partial u_i(\mathbf{a})/\partial a_i^k < 0$ for all i and all k.

[2] $\partial u_i(\mathbf{a})/\partial a_j^k \geq 0$ for all $j \neq i$ and all k.

OA1.2. **Efficiency.** The generalization of our efficiency result is as follows. Recall that, by the Perron–Frobenius theorem, any nonnegative, irreducible square matrix $\mathbf{M}$ has a left eigenvector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\mathbf{M} = r(\mathbf{M})\boldsymbol{\theta}$, where $r(\mathbf{M})$ is the spectral radius. This eigenvector is determined uniquely up to scale, and imposing the normalization that $\boldsymbol{\theta} \in \Delta_n$ (the simplex in $\mathbb{R}^n_+$) we call it *the* Perron vector of $\mathbf{M}$.

Proposition **OA1.** Consider an interior action profile $\mathbf{a} \in \mathbb{R}^{nk}_{++}$. Then the following are equivalent:

(i) $\mathbf{a}$ is Pareto efficient;
(ii) every matrix in the set $\{\mathbf{B}^{[d]}(\mathbf{a}) : d = 1, \ldots, k\}$ has spectral radius 1, and they all have the same left Perron vector.

Proof. For any nonzero $\boldsymbol{\theta} \in \Delta_n$ define $\mathcal{P}(\boldsymbol{\theta})$, the Pareto problem with weights $\boldsymbol{\theta}$ as:

$$\text{maximize} \sum_{i \in N} \theta_i u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}^{nk}_+.$$

By a standard fact, an action profile $\mathbf{a}$ is Pareto efficient if and only if it solves $\mathcal{P}(\boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \Delta_n$. The first order conditions for this problem consist of the equations $\sum_i \theta_i \partial u_i(\mathbf{a})/\partial a^d_j = 0$ for all j and all d. Rearranging, and recalling the assumption that $\partial u_j(\mathbf{a})/\partial a^d_j < 0$ for every i and d, we have:

$$\theta_j = \sum_{i \neq j} \theta_i \frac{\partial u_i(\mathbf{a})/\partial a^d_j}{-\partial u_j(\mathbf{a})/\partial a^d_j}. \tag{OA-1}$$

Given the concavity of $\mathbf{u}$, these conditions are necessary and sufficient for an interior optimum. We can summarize these conditions as the system of (matrix) equations:

$$\boldsymbol{\theta} = \boldsymbol{\theta}\mathbf{B}^{[d]}(\mathbf{a}) \quad d = 1, 2, \ldots, k. \tag{OA-2}$$

In summary, (i) is equivalent to the statement "system (OA-2) holds for some nonzero $\boldsymbol{\theta} \in \Delta_n$," and so we will treat the two interchangeably.

We can now show (i) and (ii) are equivalent. System (OA-2) holding for a nonzero $\boldsymbol{\theta} \in \Delta_n$ entails that the spectral radius of each $\mathbf{B}^{[d]}(\mathbf{a})$ is 1, because (by the Perron–Frobenius Theorem) a nonnegative eigenvector can correspond only to a largest eigenvalue. And the same system says a single $\boldsymbol{\theta}$ is a left Perron vector for each $\mathbf{B}^{[d]}(\mathbf{a})$. So (ii) holds. Conversely, if (ii) holds, then there is some left Perron vector $\boldsymbol{\theta} \in \Delta_n$ so that the system in (OA-2) holds, which (as we have observed) is equivalent to (i). □

OA1.3. **Characterizing Lindahl Outcomes.** Our characterization of Lindahl outcomes will rely on some "stacked" versions of matrices we have encountered before. We define a stacked n-by-nk Jacobian as follows:

$$\underline{\mathbf{J}}(\mathbf{a}) = \begin{bmatrix} \mathbf{J}^{[1]}(\mathbf{a}) & \mathbf{J}^{[2]}(\mathbf{a}) & \cdots & \mathbf{J}^{[k]}(\mathbf{a}) \end{bmatrix}.$$

For defining a Lindahl outcome, we will need to think of a larger price matrix. In particular, we will introduce an n-by-nk matrix

$$\underline{\mathbf{P}} = \begin{bmatrix} \mathbf{P}^{[1]} & \mathbf{P}^{[2]} & \cdots & \mathbf{P}^{[k]} \end{bmatrix},$$

where $P^{[d]}_{ij}$ (with $i \neq j$) is interpreted as the price i pays for the effort of agent j on dimension d.

To generalize our main theorem on the characterization of Lindahl outcomes, we now define a Lindahl outcome in the multi-dimensional setting. (Recall Definition 1, and from Section 4.1 that the budget balance condition can be restated as $\mathbf{Pa}^* \leq \mathbf{0}$.)

Definition **OA1.** An action profile $\mathbf{a}^*$ is a *Lindahl outcome* for a preference profile u_i if there is an n-by-nk price matrix $\underline{\mathbf{P}}$, with each column summing to zero, so that the following conditions hold for every i:

(i) The inequality

$$\underline{\mathbf{P}}\mathbf{a} \leq \mathbf{0} \tag{$\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$}$$

is satisfied when $\mathbf{a} = \mathbf{a}^*$;

(ii) for any $\mathbf{a}$ such that $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$ is satisfied, we have $\mathbf{a}^* \succeq_{u_i} \mathbf{a}$.

Definition **OA2.** The action vector $\mathbf{a} \in \mathbb{R}^{nk}_+$ is defined to be *scaling-indifferent* if $\mathbf{a} \neq \mathbf{0}$ and $\underline{\mathbf{J}}(\mathbf{a})\mathbf{a} = \mathbf{0}$.

We will establish that Lindahl outcomes are characterized by being scaling-indifferent and Pareto efficient.

Theorem **OA1.** Under the maintained assumptions, an interior action profile is a Lindahl outcome if and only if it is scaling-indifferent and Pareto efficient.

Proof. First, we show Lindahl outcomes are scaling-indifferent and Pareto efficient. Suppose $\mathbf{a}^* \in \mathbb{R}^{nk}_{++}$ is a nonzero Lindahl outcome. Its Pareto efficiency follows by the standard proof of the first welfare theorem. Let $\underline{\mathbf{P}}$ be the price matrix in Definition OA1. Consider the following program for each agent i, denoted by $\Pi_i(\underline{\mathbf{P}})$:

$$\text{maximize } u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}^{nk}_+ \text{ and } \widehat{\mathrm{BB}}_i(\underline{\mathbf{P}}).$$

By definition of a Lindahl outcome, $\mathbf{a}^*$ solves $\Pi_i(\underline{\mathbf{P}})$. By the assumption of connected benefit flows, there is always some other agent j and some dimension d so that i is better off when a_j^d increases. So the budget balance constraint $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$ is satisfied with equality. Note that this is equivalent to the statement $\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}$.

Because $\mathbf{a}^*$ is interior, the gradient of the maximand u_i (viewed as a function of $\mathbf{a}$) must be orthogonal to the budget constraint $\underline{\mathbf{P}}\mathbf{a} \leq \mathbf{0}$. In other words, row i of $\underline{\mathbf{J}}(\mathbf{a}^*)$ is parallel to row i of $\underline{\mathbf{P}}$. This combined our earlier deduction that $\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}$ implies $\underline{\mathbf{J}}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$.

We now prove the converse implication of the theorem. Take any scaling-indifferent and Pareto efficient outcome $\mathbf{a}^* \in \mathbb{R}^{nk}_+$. Because $\mathbf{a}^*$ is Pareto efficient, by Proposition OA1 there is a nonzero vector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*) = \mathbf{0}$ for each d. We need to find prices supporting $\mathbf{a}^*$ as a Lindahl outcome. Define the matrix $\mathbf{P}^{[d]}$ by $P_{ij}^{[d]} = \theta_i J_{ij}^{[d]}(\mathbf{a}^*)$ and note that for all $j \in N$ we have

$$\sum_{i \in N} P_{ij}^{[d]} = \sum_{i \in N} \theta_i J_{ij}^{[d]}(\mathbf{a}^*) = \left[\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)\right]_j = 0, \tag{OA-3}$$

where $\left[\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)\right]_j$ refers to entry j of the vector $\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)$.

Now, recalling the definition of the n-by-nk matrix $\underline{\mathbf{P}}$, we see that each column of $\underline{\mathbf{P}}$ sums to zero. Further, each row of $\underline{\mathbf{P}}$ is just a scaling of the corresponding row of $\underline{\mathbf{J}}(\mathbf{a}^*)$. We therefore have:

$$\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}, \tag{OA-4}$$

and these prices satisfy budget balance.

Finally, we claim that, for each i, the vector $\mathbf{a}^*$ solves $\Pi_i(\underline{\mathbf{P}})$. This is because the gradient of u_i at $\mathbf{a}^*$, which is row i of $\underline{\mathbf{J}}(\mathbf{a}^*)$, is normal to the constraint set by construction of $\mathbf{P}$ and, by (OA-4) above, $\mathbf{a}^*$ satisfies the constraint $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$. The claim then follows by the concavity of u_i. $\square$

OA2. Transfers of a Numeraire Good

It is natural to ask what happens in our model when transfers are possible. If utility is transferable—that is, if a "money" term enters additively into all payoffs, but utility functions are otherwise the same—then Coasian reasoning implies that the only Pareto-efficient solutions involve action profiles that maximize $\sum_i u_i(a_{\mathrm{X}}, a_{\mathrm{Y}}, a_{\mathrm{Z}})$. But in general, agents' preferences over environmental or other public goods need not be quasilinear in any numeraire—especially when the changes being contemplated are large. It is in this case that our analysis extends in an interesting way, and that is what we explore in this section, via two different modeling approaches.

OA2.1. **The Multiple Actions Approach.** We can use the extension to multiple actions to consider what will happen if we permit transfers of a numeraire good. We extend the environment in the main part of the paper by letting each agent choose, in addition to an action level, how much of a numeraire good to transfer to each other agent. We model this by assuming that each agent has $k = n$ dimensions of action. For agent i, action a_i^i corresponds to the actions we consider in the one-dimensional model of the paper and action a_i^j for $j \neq i$ corresponds to a transfer of the numeraire good from agent i to agent j. We assume agents' utility functions are concave, and that all of them always have strictly positive marginal value from consuming the numeraire good. For agent i, the transfer action a_i^j (for $j \neq i$) is then individually costly (as i can then consume less of the numeraire good) but provides weak benefits to all others. Moreover, we assume for this section that $\partial u_i / \partial a_j^j > 0$ for every i and j—meaning that the original problem has strictly positive externalities.[3] As a consequence, each $\mathbf{B}^{[d]}$ is irreducible. This extension of the single action model to accommodate transfers fits the multiple actions framework above and we can then simply apply Proposition OA1 and Theorem OA1 to show how our results change once transfers are possible.

Proposition OA1 and Theorem OA1 show that the main results of our paper extend in a natural way to environments with transfers. However, it is important to note that although we are assuming transfers are possible, we are *not* assuming that agents' preferences are quasi-linear in any numeraire. Under the (strong) additional assumption of transferable utility, the problem becomes much simpler, as mentioned above.

OA2.2. **An Inverse Marginal Utility of Money Characterization of Lindahl Outcomes with a Transferable Numeraire.** It is possible to extend Theorem 1 in a different way to a setting with a transferable, valuable numeraire. Consider the basic setting of the paper in which each player can put forth externality-generating effort on one dimension, and suppose that each agent's utility function is $u_i : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}$. We write a typical payoff as $u_i(\mathbf{a}; m_i)$, where $\mathbf{a} \in \mathbb{R}_+^n$ is an action profile as in the

[3]We suspect this assumption can be relaxed substantially without affecting the conclusions.

main text, and m_i is a net transfer of "money"—a numeraire—to agent i. We assume preferences are concave and continuously differentiable on the domain $\mathbb{R}^n_+ \times \mathbb{R}$. We also assume that for all fixed vectors $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, the utility functions satisfy the maintained assumptions of Section 2.2 in the main text. We assume that the numeraire is valuable: $\frac{\partial u_i}{\partial m_i} > 0$ on the whole domain. Finally, to streamline things, we assume that $\frac{\partial u_i}{\partial a_i}(\mathbf{a}; m_i) = -1$ for all values of $(\mathbf{a}; m_i)$. The benefits matrix is defined as in Section 2.3.

Now we can define a Lindahl outcome in this setting, taking all prices to be in terms of the numeraire.

Definition **OA3.** An outcome $(\mathbf{a}^*; \mathbf{m}^*)$ is a *Lindahl outcome* for a preference profile $\mathbf{u}$ if $\sum_{i \in N} m_i = 0$ and there is an n-by-n matrix (of prices) $\mathbf{P}$ so that the following conditions hold for every i:

(i) The inequality

$$\sum_{j:j\neq i} P_{ij} a_j + m_i \leq a_i \sum_{j:j\neq i} P_{ji} \tag{BB$_i$($\mathbf{P}$)}$$

is satisfied when $(\mathbf{a}; \mathbf{m}) = (\mathbf{a}^*; \mathbf{m}^*)$;

(ii) for any $(\mathbf{a}; m_i)$ such that the inequality BB$_i(\mathbf{P})$ is satisfied, we have

$$(\mathbf{a}^*; m_i^*) \succeq_{u_i} (\mathbf{a}; m_i).$$

We can now characterize the Lindahl outcomes in this setting in a way that is reminiscent of both Proposition 1 in Section 3 and of Theorem 1. To do this, we make one final definition.

Define

$$\mu_i(\mathbf{a}, m_i) = \left[\frac{\partial u_i}{\partial m_i}(\mathbf{a}, m_i)\right]^{-1}.$$

This is the reciprocal of i's marginal utility of the numeraire at a given outcome. We will write $\boldsymbol{\mu}(\mathbf{a}, \mathbf{m})$ for the vector of all these inverse marginal utilities.

Proposition **OA2.** An interior outcome $(\mathbf{a}; \mathbf{m})$ is a Lindahl outcome if and only if

$$\boldsymbol{\theta} = \boldsymbol{\theta}\mathbf{B}(\mathbf{a}; \mathbf{m}) \tag{OA-5}$$

where $\boldsymbol{\theta} = \boldsymbol{\mu}(\mathbf{a}, \mathbf{m})$ and

$$m_i = \theta_i \cdot \left(a_i - \sum_j B_{ij} a_j\right) \tag{OA-6}$$

for each i.

Without going through the proof, which is analogous to that of Theorem 1, we discuss the key parts of the reasoning. Given a pair $(\mathbf{a}; \mathbf{m})$ such that (OA-5) and (OA-6) hold, we will construct prices supporting $(\mathbf{a}; \mathbf{m})$ as a Lindahl outcome. For $i \neq j$, we set

$$P_{ij} = \theta_i B_{ij}(\mathbf{a}, \mathbf{m}).$$

The prices agent i faces are proportional to his marginal utilities for various other agents' contributions, so i is making optimal tradeoffs in setting the a_j for $j \neq i$. Now we turn to the "labor supply decision" of agent i, i.e., what a_i should be. The wage that i makes from working is θ_i per unit of effort, because (by equation OA-5) we can write $\sum_{j:j\neq i} P_{ji} = \sum_{j:j\neq i} \theta_j B_{ji} = \theta_i$. Thus, recalling that the price of the numeraire is 1 by definition, we have

$$\frac{\text{price of numeraire}}{i\text{'s wage}} = \frac{1}{\theta_i} = \frac{1}{[\partial u_i/\partial m_i]^{-1}} = \frac{\partial u_i/\partial m_i}{1}.$$

Recalling that 1 is the marginal disutility of effort (by assumption), this shows that the price ratio above is equal to the corresponding ratio of i's marginal utilities. Finally, the condition $m_i = \theta_i \cdot \left(a_i - \sum_j B_{ij} a_j\right)$ can be written, in terms of our prices, as

$$m_i = a_i \sum_{j:j\neq i} P_{ji} - \sum_{j:j\neq i} P_{ij} a_j.$$

In rewriting the first term, we have again used (OA-5). This equation just says that i's budget balance condition holds: The net transfer of the numeraire he obtains is the difference between the wages paid to him and what he owes others for their contributions.

This shows that the conditions of Proposition OA2 are sufficient for a Lindahl outcome. The omitted argument for the converse is simpler; the proof essentially involves tracing backward through the reasoning we have just given.

The important thing to note about the conditions of Proposition OA2 is that, like the characterization of Theorem 1, there are no prices explicitly involved. The content of the Lindahl solution can be summarized succinctly in an eigenvector equation. Here the equation says that an agent's θ_i, his inverse marginal utility of income (so a higher θ_i corresponds to more wealth), satisfies the eigenvector centrality equation $\theta_i = \sum_j B_{ji}\theta_j$. Equivalently, the θ_i's are proportional to agents' eigenvector centralities in the network $\mathbf{B}(\mathbf{a})^\mathsf{T}$. Using the walks interpretation discussed in Section 5, we can say the following: In the presence of transfers, wealthier (higher θ_i) agents are the ones who sit at the *origin* of large flows in the benefits matrix: They are the ones capable of conferring large direct and indirect benefits on others.

OA3. A Group Bargaining Foundation for the Lindahl Solution

In Section 4.2.1 we argue that the Lindahl solution can be motivated as the equilibrium outcome of a group bargaining game. In this section we flesh out those claims more precisely.

The bargaining game begins in state s_0, and the timing of the game within a period is:

(i) A new proposer is selected according to a stationary, irreducible Markov chain on N
(ii) The proposer $\nu(s)$ selects a direction $\mathbf{d} \in \Delta^n$, where Δ^n is the simplex in $\mathbb{R}^n$.
(iii) All agents then simultaneously respond. Each may vote "no" *or* may specify a maximum scaling of the proposed direction by selecting $\lambda_i \in \mathbb{R}_+$.
(iv) If any agent votes "no", the proposal is rejected and we return to step (i), in which someone else is selected to propose a direction.
(v) If nobody votes "no", then actions $\mathbf{a} = (\min_i \lambda_i)\mathbf{d}$ are implemented.

The game can go on for infinitely many periods. Until an agreement is reached and actions are taken, players receive their status quo payoffs $u_i(\mathbf{0}) = \mathbf{0}$ each period; afterward they receive the payoffs of the implemented action forever. Players evaluate streams of payoffs according to the expectation of a discounted sum of period payoffs. We fix a common discount factor $\delta \in (0, 1)$.

We will show that efficient outcomes are obtainable in equilibrium and we will characterize this set. More precisely, we will find the set of efficient perfect equilibrium outcomes in this game—i.e., ones resulting in paths of play not Pareto dominated by any other path of play.[4] Let $A(\delta)$ be the set of nonzero action profiles $\mathbf{a}$ played in some efficient perfect equilibrium for discount factor δ.

Proposition **OA3.** Suppose actions $\mathbf{a} = \mathbf{0}$ are Pareto inefficient, that utilities are strictly concave, and that the assumptions of Section 2.2 hold. Then $A(\delta)$ is the set of Lindahl outcomes—or, equivalently, the set of centrality action profiles.

Before presenting the proof, we outline the main ideas of the argument here.[5] First, note that Pareto efficiency requires that, in every state, the same deterministic action profile be agreed on during the first round of negotiations.[6] Delay is inefficient as there is discounting, and the strict concavity of utility functions means that it is also inefficient for different actions to be played with positive probability—it would be a Pareto improvement to play a convex combination of those actions instead. Consider now which deterministic actions can be played. Intuitively, the structure of the game can be interpreted as giving all agents veto power over how far actions are scaled up in the proposed direction. This constrains the possible equilibrium outcomes to those in which no agent would want to scale down actions. Next, we show that if there are some agents who strictly prefer to scale *up* actions at the margin, while all other agents are (first-order) indifferent, the current action profile is Pareto inefficient. The set of action profiles that remain as candidate efficient equilibrium outcomes are those in which all agents are indifferent to scaling the actions up *or* down at the margin. Recalling Definition 3 in Section 4.1, these are the centrality action profiles. This is why only centrality action profiles can occur in an efficient perfect equilibrium. The proof is completed by constructing such an equilibrium for any centrality action profile.

Proof of Proposition OA3: We begin by showing that in all Pareto efficient perfect equilibria, a centrality action profile must be played.

Pareto efficiency requires two things. First, as there is discounting ($\delta < 1$), it requires that that agreement be reached at the first round of negotiations. Second,

[4]There will also be many inefficient equilibria. For example, for any direction, it is an equilibrium in the second stage of the game for all agents to select the zero action profile, as none of them will be pivotal when they do so. Requiring efficiency rules out these equilibria, but perhaps more reasonable equilibria too.

[5]Penta (2011) has a similar result in which the equilibria of games without externalities converge to the Walrasian equilibria as players become patient. As we saw in Section 4, Walrasian equilibria are closely related to our eigenvector centrality condition. Nevertheless, the settings are quite different. Penta (2011) considers an endowment economy, and it is important for his results that, whenever outcomes are Pareto inefficient, there is a pair of agents that can find a profitable pairwise trade. This does not hold in our framework.

[6]As defined above, our notion of Pareto efficiency requires that no sequence of action profiles can be found that yields a Pareto improvement from the *ex ante* perspective. This notion is quite strong, as the first step in the argument demonstrates. We could instead require only that, in each period, a Pareto efficient action profile be played. Under this weaker condition, we conjecture a version of Proposition OA3 holds in the limit as $\delta \to 1$.

Pareto efficiency requires that, almost surely, some particular action profile be played on the equilibrium path, regardless of the state reached in the first period. Toward a contradiction, suppose there is immediate agreement but that different agreements are reached in different states that occur with positive probability. Let $\mathbf{a}(s)$ be the actions played in equilibrium in state s. The probability of being in state s for the first round of negotiations is $p(s_0, s)$. As utility functions are strictly concave, a Pareto improvement can be obtained by the players choosing strategies that result in the deterministic action profile $\overline{\mathbf{a}} = \sum_{s \in S} p(s_0, s)\mathbf{a}(s)$ being played in all states.

So let $\mathbf{a}$ be the nonrandom Pareto efficient action profile on which players immediately agree in some efficient perfect equilibrium of the game. We will show it is a centrality action profile. If $\mathbf{J}(\mathbf{a})\mathbf{a}$ has a negative entry, say i, then player i did not best-respond in stage (iii) of the game, in which a scaling was selected. By choosing a smaller λ_i (for example, the largest λ_i such that $[\mathbf{J}(\lambda_i \mathbf{d})\mathbf{d}]_i \geq 0$), that player would have secured a strictly higher payoff.

Therefore, $\mathbf{J}(\mathbf{a})\mathbf{a} \geq \mathbf{0}$. We claim this holds with equality. Suppose, by way of contradiction, that it does not. Then

$$\mathbf{D}(\mathbf{a})^{-1}\mathbf{J}(\mathbf{a})\mathbf{a} \gneqq \mathbf{0},$$

where $\mathbf{D}(\mathbf{a})$ is a diagonal matrix with $D_{ii}(\mathbf{a}) = -J_{ii}(\mathbf{a})$ and zeros off the diagonal. We then have $\left(\mathbf{D}(\mathbf{a})^{-1}\mathbf{J}(\mathbf{a}) + \mathbf{I}\right)\mathbf{a} = \mathbf{B}(\mathbf{a})\mathbf{a} \gneqq \mathbf{a}$. By irreducibility of $\mathbf{B}(\mathbf{a})$, there then exists an $\mathbf{a}'$ such that $\mathbf{B}(\mathbf{a})\mathbf{a}' > \mathbf{a}'$ (with strict inequalities in each entry). The Collatz–Wielandt formula (Meyer, 2000, equation 8.3.3) says that $r(\mathbf{B}(\mathbf{a}))$ is given by:

$$\min_{a'_i} \frac{[\mathbf{B}(\mathbf{a})\mathbf{a}']_i}{a'_i}.$$

Thus, $r(\mathbf{B}(\mathbf{a})) > 1$ and, by Proposition 1, $\mathbf{a}$ is Pareto inefficient, which is a contradiction.

Thus, we have established that $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$. Because the action profile $\mathbf{0}$ is not Pareto efficient by assumption, we deduce that $\mathbf{a}$ is nonzero, and therefore it is scaling-indifferent. Applying the definition of $\mathbf{B}$, we conclude $\mathbf{a}$ is a centrality action profile.

To finish the proof, it remains only to show that for any centrality action profile $\mathbf{a}$, we can find a perfect equilibrium that supports it. The strategies are as follows: Any player, when proposing a direction, suggests $\mathbf{d} = \mathbf{a}/\sum_i a_i$, i.e., the normalization of $\mathbf{a}$. When responding to proposals, every player vetoes any direction other than $\mathbf{d}$. On the other hand, if $\mathbf{d}$ is proposed, then player i sets

$$\lambda_i = \min\{\lambda : [\mathbf{J}(\lambda \mathbf{d})\mathbf{d}]_i \leq 0\}. \tag{OA-7}$$

This is well-defined because for $\lambda = \sum_i a_i$, we have $\mathbf{J}(\lambda \mathbf{d})\mathbf{d} = \lambda^{-1}\mathbf{J}(\mathbf{a})\mathbf{a}$, all of whose entries are 0 because $\mathbf{a}$ is scaling-indifferent. Indeed, by strict concavity of the utility functions, $[\mathbf{J}(\lambda \mathbf{d})\mathbf{d}]_i$ is decreasing in λ and so $\lambda_i = \sum_i a_i$ for all i. Thus direction $\mathbf{d}$ is proposed and actions $\lambda \mathbf{d} = \mathbf{a}$ are selected under this strategy profile.

The proof that this is an equilibrium is straightforward. Consider i's incentives. Given that the other players respond to proposals as specified in this strategy profile, the only outcomes that can ever be implemented are in the set $P = \{\mu \mathbf{d} : 0 \leq \mu \leq \max_{j \neq i} \lambda_j\}$. Consider a subgame where someone has proposed direction $\mathbf{d}$. Voting "no" can yield only some action in P at a later period (or no agreement forever). By definition of λ_i, responding with λ_i yields maximum utility among all points in P; thus, players have incentives to follow the strategy profile when responding to a proposal of direction $\mathbf{d}$. The same argument shows that proposing a direction other than $\mathbf{d}$ cannot be a profitable deviation—it will result in rejection and the implementation of something in P later—whereas by playing the proposed equilibrium, i could obtain the payoff of $\mathbf{a}$ now. Finally, when a direction other than $\mathbf{d}$ is proposed, players are indifferent between voting "yes" and voting "no", because the proposal will be rejected by the votes of the others. ∎

OA4. Implementation Theory Foundations for the Lindahl Solution: Formal Details

Section 4.2.2 discussed the unique robustness of Lindahl outcomes from the perspective of a mechanism design problem. In this section, we present the notation and results to make that discussion fully precise.

Let $\mathcal{U}_A$ be the set of all functions $u : \mathbb{R}^n_+ \to \mathbb{R}$. We denote by $\succeq_u$ and $\succ_u$ the weak and strict preference orderings, respectively, induced by $u \in \mathcal{U}_A$. The domain of possible preference profiles[7] is a set $\mathcal{U} \subseteq \mathcal{U}_A^n$; we will state specific assumptions on it in our results.

A *game form* is a tuple $H = (\Sigma_1, \ldots, \Sigma_n, g)$ where:

- Σ_i is a set of *strategies* that agent i can play; we write $\Sigma = \prod_{i \in N} \Sigma_i$;
- $g : \Sigma \to \mathbb{R}^n_+$ is the *outcome function* that maps strategy profiles to action profiles.

Definition **OA4.** In a game form $H = (\Sigma_1, \ldots, \Sigma_n, g)$, a strategy profile $\boldsymbol{\sigma} \in \Sigma$ is a *Nash equilibrium* for preference profile $\mathbf{u} \in \mathcal{U}$ if for any $i \in N$ and any $\widetilde{\sigma}_i \in \Sigma_i$, it holds that $g(\boldsymbol{\sigma}) \succeq_{u_i} g(\widetilde{\sigma}_i, \boldsymbol{\sigma}_{-i})$. We define $\Sigma^*(H, \mathbf{u})$ to be the set of all such $\boldsymbol{\sigma}$.

A *social choice correspondence* $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ maps each preference profile to a nonempty set of outcomes. Any game form for which equilibrium existence is guaranteed[8] naturally induces a social choice correspondence: its Nash equilibrium outcome correspondence $F_H(\mathbf{u}) = g(\Sigma^*(H, \mathbf{u}))$. The set $F_H(\mathbf{u})$ describes all the outcomes the participants with preferences $\mathbf{u}$ can end up with if they are left with a game form H and they play some Nash equilibrium. We say that F_H is the social choice correspondence that the game form H *implements*[9]. A social choice correspondence is said to be *implementable* if there is some game form H that implements it.

There are two basic normative criteria we impose on such correspondences. A social choice correspondence F is *Pareto efficient* if, for any $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a} \in F(\mathbf{u})$, the profile $\mathbf{a}$ is Pareto efficient under $\mathbf{u}$. A social choice correspondence F is *individually rational* if, for any $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a} \in F(\mathbf{u})$, it holds that $\mathbf{a} \succeq_{u_i} \mathbf{0}$ for all i. An individually rational social choice correspondence is one that leaves every player no worse off than the status quo.

We will also refer to a technical condition—upper hemicontinuity. A social choice correspondence F is *upper hemicontinuous* if: For every sequence of preference profiles $(\mathbf{u}^{(k)})$ converging compactly[10] to $\mathbf{u}$, and every sequence of outcomes $\left(\mathbf{a}^{(k)}\right)$ with $\mathbf{a}^{(k)} \in$

[7] The standard approach (e.g., Maskin, 1999) is to work with preference relations. We use sets of utility functions to avoid carrying around two parallel notations.

[8] Otherwise, we can still talk about the correspondence, but it will not be a social choice correspondence, which is required to be nonempty-valued.

[9] To be more precise, this is the definition of full Nash implementation. Since we consider only this kind of implementation, we drop the adjectives.

[10] That is, the sequence $(\mathbf{u}^{(k)})$ converges uniformly on every compact set.

$F(\mathbf{u}^{(k)})$, if $\mathbf{a}^{(k)} \to \mathbf{a}$, then $\mathbf{a} \in F(\mathbf{u})$. This condition has some normative appeal in that a social choice correspondence not satisfying upper hemicontinuity is sensitive to arbitrarily small changes in preferences that may be difficult for the agents themselves to detect.[11]

Definition **OA5.** The *Lindahl correspondence* $L : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ is defined by

$$L(\mathbf{u}) = \{\mathbf{a} \in \mathbb{R}^n_+ : \mathbf{a} \text{ is a Lindahl outcome for } \mathbf{u}\}.$$

Fix $\mathcal{U}$. Let $\mathcal{F}$ be the set of implementable social choice correspondences $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ that are Pareto efficient, individually rational, and upper hemicontinuous. For any $\mathbf{u} \in \mathcal{U}$, define the set of outcomes prescribed at $\mathbf{u}$ by *every* such correspondence:

$$R(\mathbf{u}) = \bigcap_{F \in \mathcal{F}} F(\mathbf{u}). \tag{OA-8}$$

This defines a correspondence $R : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$. We call this the *robustly attainable* correspondence.

If the set of possible preferences is rich enough, then the robustly attainable correspondence is precisely the Lindahl correspondence. We can now formally state the result mentioned in Section 4.2.2.

Proposition **OA4.** Suppose $\mathcal{U}$ is the set of all preference profiles satisfying the assumptions of Section 2.2, and the number of players n is at least 3. Then the robustly attainable correspondence is equal to the Lindahl correspondence: $R = L$.

From this proposition, we can deduce that the Lindahl correspondence is the *minimum* solution in $\mathcal{F}$—it is the unique one that is a subcorrespondence of every other. For details on this, see Section OA4.2 below.

OA4.1. **Proof.** We begin by recalling Maskin's Theorem. Assuming that the number of agents n is at least 3 and that a social choice correspondence F satisfies *no veto power*[12] (a condition that is vacuously satisfied in our setting), then F is implementable if and only if it satisfies *Maskin monotonicity.*

Definition **OA6.** A social choice correspondence $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ satisfies *Maskin monotonicity* if: Whenever $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$ and for some $\mathbf{u} \in \mathcal{U}$ it holds that

$$\forall i \in N, \forall \mathbf{a} \in \mathbb{R}^n_+, \quad \mathbf{a}^* \succeq_{\widehat{u}_i} \mathbf{a} \;\Rightarrow\; \mathbf{a}^* \succeq_{u_i} \mathbf{a}, \tag{OA-9}$$

then $\mathbf{a}^* \in F(\mathbf{u})$.[13]

We now show that[14] $R \subseteq L$. By the definition that $R(\mathbf{u}) = \bigcap_{F \in \mathcal{F}} F(\mathbf{u})$, it suffices to show that $L \in \mathcal{F}$, i.e., that L is an implementable, individually rational, Pareto efficient, and upper hemicontinuous social choice correspondence. First, a social choice correspondence must be nonempty-valued; Proposition 2 in Section 4.2.1 guarantees that L complies. By Assumption 3, the no veto power condition is vacuous in our

[11]The other way for upper hemicontinuity to fail is for the values of F not to be closed sets.

[12]A social choice correspondence $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ satisfies no veto power if, for every $\mathbf{u} \in \mathcal{U}$, whenever there is an $\mathbf{a} \in \mathbb{R}^n_+$ and an agent i' such that $\mathbf{a} \succeq_{u_i} \mathbf{a}'$ for all $i \neq i'$ and all $\mathbf{a}' \in \mathbb{R}^n_+$, then $\mathbf{a} \in F(\mathbf{u})$.

[13]In words: If an alternative $\mathbf{a}^*$ was selected by F under $\widehat{\mathbf{u}}$ and then we change those preferences to a profile $\mathbf{u}$ so that (under each agent's preference) the outcome $\mathbf{a}^*$ defeats all the same alternatives that it defeated under $\widehat{\mathbf{u}}$ and perhaps some others, then $\mathbf{a}^*$ is still selected under $\mathbf{u}$.

[14]For two correspondences $F, F^\ddagger : \mathcal{U} \to \mathbb{R}^n_+$, we write $F \subseteq F^\ddagger$ if for every $\mathbf{u} \in \mathcal{U}$, it holds that $F(\mathbf{u}) \subseteq F^\ddagger(\mathbf{u})$. In this case, we say that F is a *sub-correspondence* of $F^\ddagger$.

setting. It is verified immediately from Definition 1 that L satisfies Maskin monotonicity.[15] Thus, L is implementable by Maskin's Theorem. Also, L is individually rational since, by definition of a Lindahl outcome, each agent prefers a Lindahl outcome to $\mathbf{0}$, which is always feasible. By the standard proof of the First Welfare Theorem, L is Pareto efficient (see, e.g., Foley, 1970). Similarly, the standard argument for the upper hemicontinuity of equilibria in preferences transfers to our setting.

Now assume F is implementable, Pareto efficient, individually rational, and upper hemicontinuous. Fix $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a}^* \in L(\mathbf{u})$. We will show $\mathbf{a}^* \in F(\mathbf{u})$. Define

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a}.$$

Lemma OA1, proved later in this section, states that since F is individually rational, Pareto efficient, and upper hemicontinuous, it follows that $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.[16] Note that for all $\mathbf{a} \in \mathbb{R}^n_+$, we have

$$\widehat{\mathbf{u}}(\mathbf{a}^*) - \widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})(\mathbf{a}^* - \mathbf{a}) \leq \mathbf{u}(\mathbf{a}^*) - \mathbf{u}(\mathbf{a})$$

by concavity of $\mathbf{u}$, so (OA-9) holds. Since F is implementable, it satisfies Maskin monotonicity, so we conclude that $\mathbf{a}^* \in F(\mathbf{u})$. ∎

The Hurwicz rationale for the Lindahl outcomes is actually more general than we have so far stated. We will now formalize and prove this.

Let $\mathcal{A}$ be the set of preference profiles $\mathbf{u}$ satisfying the assumptions of Section 2.2. Endow this space with the compact-open topology.[17]

Definition **OA7.** A set of preferences $\mathcal{U} \subseteq \mathcal{A}$ is called *rich* if, for every $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a}^* \in \mathbb{R}^n_+$, there is a (linear) preference profile $\widehat{\mathbf{u}} \in \mathcal{U}$ defined by

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a} \tag{OA-10}$$

and a neighborhood of $\widehat{\mathbf{u}}$ relative to $\mathcal{A}$ is contained in $\mathcal{U}$.

Richness of $\mathcal{U}$ requires that for every preference profile $\mathbf{u} \in \mathcal{U}$ and every $\mathbf{a}^* \in \mathbb{R}^n_+$, there are preferences in $\mathcal{U}$ that are linear over outcomes and have the same *marginal* tradeoffs that $\mathbf{u}$ does at $\mathbf{a}^*$, as well as a neighborhood of these preferences. To take a simple example, $\mathcal{A}$ itself is rich.

Proposition **OA5.** Suppose $\mathcal{U}$ is rich and the number of players, n, is at least 3. Then the robustly attainable correspondence is equal to the Lindahl correspondence: $R = L$.

The proof is exactly as in Section OA4. The only thing that remains to do is to establish the following lemma used in that proof under the hypothesis that $\mathcal{U}$ is rich (the result needed in Section OA4 is then a special case).

[15] If $\widehat{\mathbf{u}}$ and $\mathbf{u}$ are as in the above definition of Maskin monotonicity and $\mathbf{a}$ is a Lindahl outcome under preferences $\widehat{\mathbf{u}}$, then using the same price matrix $\mathbf{P}$, the outcome $\mathbf{a}$ still satisfies condition (ii) in Definition 1.

[16] The proof of that lemma constructs a sequence of preference profiles $(\widehat{\mathbf{u}}^{(k)})$ converging to $\widehat{\mathbf{u}}$ such that individual rationality and Pareto efficiency alone force the set $F(\widehat{\mathbf{u}}^{(k)})$ to converge to $\mathbf{a}^*$. Then by upper hemicontinuity of F, it follows that $F(\widehat{\mathbf{u}})$ contains $\mathbf{a}^*$.

[17] For any compact set $K \subseteq \mathbb{R}^n_+$ and open set $V \subseteq \mathbb{R}^n$, let $U(K, V)$ be the set of all preference profiles $\mathbf{u} \in \mathcal{A}$ so that $\mathbf{u}(K) \subseteq V$. The compact-open topology is the smallest one containing all such $U(K, V)$.

LEMMA **OA1.** Fix $\mathbf{u}$ satisfying the assumptions of Section 2.2 and an $\mathbf{a}^* \in L(\mathbf{u})$. Define $\widehat{\mathbf{u}}$ as in (OA-10), i.e.,

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a}.$$

Suppose $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ is a Pareto efficient, individually rational, and upper hemicontinuous social choice correspondence. If $\mathcal{U}$ is rich, then $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.

Proof of Lemma OA1: First assume $\mathbf{a}^* \neq \mathbf{0}$. (We will handle the other case at the end of the proof.) By Lemma 2 in Section C, $\mathbf{a}^*$ is interior—all its entries are positive. Write $\mathbf{J}^*$ for $\mathbf{J}(\mathbf{a}^*; \mathbf{u})$ and $\mathbf{B}^*$ for $\mathbf{B}(\mathbf{a}^*; \mathbf{u})$.

For $\gamma > 0$, and $i \in N$, define $\widehat{u}_i^{[\gamma]} : \mathbb{R}^n_+ \to \mathbb{R}$ by

$$\widehat{u}_i^{[\gamma]}(\mathbf{a}) = J^*_{ii} \left(\gamma + a_i\right)^{1+\gamma} + \sum_{j \neq i} J^*_{ij} a_j.$$

This is just an adjustment obtained from $\widehat{\mathbf{u}} = \widehat{\mathbf{u}}^{[0]}$ by building some convexity into the costs. Note that for all γ close enough to 0, the profile $\widehat{\mathbf{u}}^{[\gamma]}$ is in $\mathcal{U}$ by the richness assumption.[18]

Choose $\mathbf{a}^{[k]} \in F(\widehat{\mathbf{u}}^{[1/k]})$; this is legitimate since F is a social choice correspondence, and hence nonempty-valued. We will show that by the properties of F, a subsequence of the sequence $(\mathbf{a}^{[k]})$ converges to $\mathbf{a}^*$. Then by upper hemicontinuity of F, it will follow that $\mathbf{a}^* \in F(\widehat{\mathbf{u}}^{[0]})$, as desired. The trickiest part of the argument is showing that the $\mathbf{a}^{[k]}$ lie in some compact set, so we can extract a convergent subsequence; it will then be fairly easy to show that the limit point of that subsequence is $\mathbf{a}^*$.

Let $\mathrm{IR}^{[\gamma]}$ be the set of individually rational points under $\widehat{\mathbf{u}}^{[\gamma]}$, and let $\mathrm{PE}^{[\gamma]}$ be the set of Pareto efficient points under $\widehat{\mathbf{u}}^{[\gamma]}$. Let $a^*_{\max} = \max_i a^*_i$, and define the box $K = [0, 2a^*_{\max}]^n$.

CLAIM **OA1.** For all k, the point $\mathbf{a}^{[k]}$ is either in K or on the ray

$$Z = \{\mathbf{a} \in \mathbb{R}^n_+ : \mathbf{J}^*\mathbf{a} = \mathbf{0}\}.$$

To show the claim, we first establish that

$$\mathrm{IR}^{[0]} = Z.$$

The proof is as follows: First note that $\widehat{\mathbf{u}}^{[0]}(\mathbf{a}) = \mathbf{J}^*\mathbf{a}$. There cannot be an $\mathbf{a}$ such that $\mathbf{J}^*\mathbf{a}$ is nonnegative in all entries and positive in some entries.[19] Thus, if $\mathbf{J}^*\mathbf{a}$ is nonzero, it must have some negative entries, i.e., $\widehat{u}_i^{[0]}(\mathbf{a}) < 0$ for some i, and then $\mathbf{a} \notin \mathrm{IR}^{[0]}$, contradicting the fact that F is individually rational.

Next, it can be seen that for $\mathbf{a}$ outside the box K, we have for small enough γ

$$\widehat{\mathbf{u}}^{[\gamma]}(\mathbf{a}) \leq \widehat{\mathbf{u}}^{[0]}(\mathbf{a}).$$

From this and the fact that $\widehat{\mathbf{u}}^{[\gamma]}(\mathbf{0}) = \mathbf{0}$ for all γ, we have the relation

$$\mathrm{IR}^{[\gamma]} \cap K^c \subseteq \mathrm{IR}^{[0]} \cap K^c.$$

---

[18]The key fact here is that the topology of compact convergence is the same as the compact-open topology (Bourbaki, 1989, Chapter X, §3.4). As $\gamma \downarrow 0$, the functions $\mathbf{u}^{[\gamma]}$ converge compactly to $\widehat{\mathbf{u}}$, and thus any neighborhood of $\widehat{\mathbf{u}}$ under the compact-open topology contains $\mathbf{u}^{[\gamma]}$ for sufficiently small $\gamma > 0$. Therefore $\mathcal{U}$ contains these functions as well (recall the definition of richness).

[19]Otherwise, $\mathbf{a}^*$ would not have been Pareto efficient under $\mathbf{u}$: moving in the direction $\mathbf{a}$ would have yielded a Pareto improvement. But $\mathbf{a}^*$ *is* Pareto efficient—see Section 4.

Since we have established that $\mathrm{IR}^{[0]} = Z$, the claim follows.

We now deduce that, in fact, $\mathbf{a}^{[k]} \in K$ for all k. It is easily checked[20] that if $\mathbf{a} \in Z$ and $\mathbf{a} > \mathbf{a}^*$, then for $\gamma > 0$ we have $r(\mathbf{B}(\mathbf{a}; \widehat{\mathbf{u}}^{[\gamma]})) < r(\mathbf{B}^*) = 1$, where the latter equality holds by the efficiency of centrality action profiles. Therefore, by Proposition 1, no point on the ray Z outside K is Pareto efficient for $\gamma > 0$. This combined with Claim OA1 shows that $\mathrm{IR}^{[\gamma]} \cap \mathrm{PE}^{[\gamma]} \subseteq K$, and therefore (since F is Pareto efficient and individually rational) it follows that $\mathbf{a}^{[k]} \in K$ for all k.

As a result we can find a sequence $(j(k))_k$ such that the sequence $\left(\mathbf{a}^{(j(k))}\right)_k$ converges to some $\overline{\mathbf{a}} \in \mathbb{R}^n_+$. Define $\mathbf{a}^{(k)} = \mathbf{a}^{[j(k)]}$ and set $\widehat{\mathbf{u}}^{(k)} = \widehat{\mathbf{u}}^{[1/j(k)]}$. Note that the $\widehat{\mathbf{u}}^{(k)}$ converge uniformly to $\widehat{\mathbf{u}}^{[0]}$ on K and, indeed, on any compact set (thus, they converge compactly to $\widehat{\mathbf{u}}^{[0]}$). By upper hemicontinuity of F, it follows that $\overline{\mathbf{a}} \in F(\widehat{\mathbf{u}}^{[0]})$. It remains only to show that $\overline{\mathbf{a}} = \mathbf{a}^*$, which we now do.

If $\overline{\mathbf{a}} \notin Z$, then it is easy to see that for large enough k, we would have $\widehat{u}_i^{(k)}(\mathbf{a}^{(k)}) < 0$ for some i. This would contradict the hypothesis that F is individually rational. Thus, $\mathbf{a}^{(k)} \to \zeta \mathbf{a}^*$ for some $\zeta \geq 0$. If $\zeta = 0$, then eventually $\mathbf{a}^{(k)}$ is not Pareto efficient for preferences $\mathbf{u}^{(k)}$, because $(\gamma + a_i)^{1+\gamma}$ with $a_i = 0$ tends to zero as $\gamma \downarrow 0$, making increases in action arbitrarily cheap (while marginal benefits remain constant). But that contradicts the Pareto efficiency of F. So assume $\zeta > 0$. In that case we would have:

$$J_{ij}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)}) \to \begin{cases} \zeta J^*_{ij} & \text{if } j = i \\ J^*_{ij} & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)}) \to \zeta \mathbf{B}^*.$$

Recall from Section 4 that $r(\mathbf{B}^*) = 1$. Since the spectral radius is linear in scaling the matrix and continuous in matrix entries, it follows that

$$r(\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)})) \to \zeta,$$

By the Pareto efficiency of F, we know that $r(\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)})) = 1$ whenever $\mathbf{a}^{(k)}$ is interior, which holds for all large enough k since $\zeta \neq 0$. Thus $\zeta = 1$. It follows that $\overline{\mathbf{a}} = \mathbf{a}^*$ and the argument is complete.

It remains to discuss the case that $\mathbf{a}^* = \mathbf{0}$ is a Lindahl outcome. In that case, by Proposition 7 in Section D (or simply the First Welfare Theorem), the outcome $\mathbf{0}$ is Pareto efficient. It follows that there cannot be any $\mathbf{a} \in \mathbb{R}^n_+$ such that $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}$ is nonzero and nonnegative; for if there were, we would be able to find a (nearby) Pareto improvement on $\mathbf{0}$ under $\mathbf{u}$. There are thus two cases: (i) $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}$ has at least one negative entry for every nonzero $\mathbf{a} \in \mathbb{R}^n_+$; or (ii) there is some nonzero $\mathbf{a}^{**} \in \mathbb{R}^n_+$ such that $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{0}$.

In case (i), it follows by concavity of $\mathbf{u}$ that $\mathbf{0}$ is the only individually rational and Pareto efficient outcome under $\widehat{\mathbf{u}}$. So $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.

In case (ii), $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{0}$ can be rewritten as $\mathbf{B}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{a}^{**}$. The Perron-Frobenius Theorem implies that $\mathbf{a}^{**}$ has only positive entries (because it is a right

[20]We do a very similar calculation below in this proof.

eigenvector of $\mathbf{B}(\mathbf{0};\mathbf{u})$, which is nonnegative and irreducible by our maintained assumptions). Now, recall the argument we carried through above in the case $\mathbf{a}^* \neq \mathbf{0}$, involving a sequence of utility functions converging to $\widehat{\mathbf{u}}$. This argument goes through without change if we replace all instances of $\mathbf{J}^*$ by $\mathbf{J}(\mathbf{0};\mathbf{u})$; all instances of $\mathbf{B}^*$ by $\mathbf{B}(\mathbf{0};\mathbf{u})$; and if we redefine[21] $\mathbf{a}^* = \beta\mathbf{a}^{**}$ for any $\beta > 0$. That shows that $\beta\mathbf{a}^{**} \in F(\widehat{\mathbf{u}})$ for every $\beta > 0$. Now, since F is an upper hemi-continuous correspondence, its values are closed: in particular, the set $F(\widehat{\mathbf{u}})$ is closed. So $\mathbf{0} \in F(\widehat{\mathbf{u}})$ as well, completing the proof.

OA4.2. **The Lindahl Correspondence as the Smallest Solution Satisfying the Desiderata.** In Section OA4, we defined a set $\mathcal{F}$ of solutions having some desirable properties (those that are Pareto efficient, individually rational, and upper hemicontinuous) and showed that the Lindahl correspondence satisfies $L(\mathbf{u}) = \bigcap_{F\in\mathcal{F}} F(\mathbf{u})$. After stating that result in Proposition OA4, we claimed that this implies that L is the unique minimum correspondence in $\mathcal{F}$. In this section, we supply the details to make that statement precise, and contrast the notion of a minimum solution with the weaker notion of a minimal one.

Let $\mathcal{U}$ be a set of problems or environments (in our case, preference profiles) and let X be a set of available allocations (in our case, action profiles in $\mathbb{R}^n_+$). Fix a particular set $\mathcal{F}$ of nonempty-valued correspondences $F : \mathcal{U} \rightrightarrows X$.[22] Given $F, G \in \mathcal{F}$, recall that we say $F = G$ if $F(\mathbf{u}) = G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$.

Definition **OA8.** An $F \in \mathcal{F}$ is a *minimum in* $\mathcal{F}$ if: for every $G \in \mathcal{F}$ and every $\mathbf{u} \in \mathcal{U}$, we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$.

This differs from the definition of a *minimal* social choice correspondence:

Definition **OA9.** An $F \in \mathcal{F}$ is *minimal in* $\mathcal{F}$ if: there is no $G \in \mathcal{F}$ satisfying $G(\mathbf{u}) \subseteq F(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$, with strict containment for some $\mathbf{u} \in \mathcal{U}$.

Minimal correspondences exist under fairly general conditions (of the Zorn's Lemma type); the existence of a minimum is a more stringent condition.[23] However, what the minimum lacks in general existence results it makes up for in uniqueness in the cases where it does exist. When a minimum exists, it is uniquely determined. (In contrast, there may in general be multiple correspondences that are minimal in $\mathcal{F}$.)

Proposition **OA6.** If each of F and G is a minimum in $\mathcal{F}$, then $F = G$.

Proof of Proposition OA6: Take any $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum in $\mathcal{F}$, we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$. By definition of G being a minimum in $\mathcal{F}$, we have $G(\mathbf{u}) \subseteq F(\mathbf{u})$. Thus $F(\mathbf{u}) = G(\mathbf{u})$. Since $\mathbf{u}$ was arbitrary, this establishes the equality. ∎

[21]Except as the argument in the definitions of $\mathbf{J}^*$ or $\mathbf{B}^*$.

[22]In our case, these are the Nash-implementable, upper hemi-continuous correspondences F so that, for each $\mathbf{u} \in \mathcal{U}$, the set $F(\mathbf{u})$ contains only Pareto efficient outcomes that leave nobody worse off than the endowment. But nothing in the present section relies on this structure.

[23]Suppose F is a minimum in $\mathcal{F}$. We will show it is minimal in $\mathcal{F}$. Suppose we have $G \in \mathcal{F}$ such that $G(\mathbf{u}) \subseteq F(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum it is also the case that $F(\mathbf{u}) \subseteq G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$. Thus $G = F$ and it is impossible for $G(\mathbf{u})$ to be strictly smaller than $F(\mathbf{u})$, for any $\mathbf{u}$. So F is, indeed, minimal. In particular, existence of a minimum in $\mathcal{F}$ implies existence of a minimal correspondence in $\mathcal{F}$. The converse does not hold.

We can give a more "constructive" characterization of the minimum that connects it with our discussion in Section OA4.

Proposition **OA7.** If F is a minimum in $\mathcal{F}$, then $F(\mathbf{u}) = \bigcap_{G\in\mathcal{F}} G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$.

Proof of Proposition OA7: Define the correspondence $H : \mathcal{U} \rightrightarrows X$ by $H(\mathbf{u}) = \bigcap_{G\in\mathcal{F}} G(\mathbf{u})$. (At this point nothing is claimed about whether H is in $\mathcal{F}$.) Now take any $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum in $\mathcal{F}$, for every $G \in \mathcal{F}$ we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$. Thus, $F(\mathbf{u})$ lies in the intersection of all the sets $G(\mathbf{u})$: that is, $F(\mathbf{u}) \subseteq H(\mathbf{u})$. On the other hand, since $F \in \mathcal{F}$ is one of the correspondences over which the intersection $\bigcap_{G\in\mathcal{F}} G(\mathbf{u})$ is taken, we have the reverse inclusion $H(\mathbf{u}) \subseteq F(\mathbf{u})$. Since $\mathbf{u}$ was arbitrary, we have shown[24] $F = H$. ∎

OA5. The Lindahl Solution and Coalitional Deviations: A Core Property

Section 4.2.3 argues that the Lindahl solution is robust to coalitional deviations in a certain sense. In this section we make those claims precise.

Formally, we make the following definition.

Definition **OA10.** An action profile $\mathbf{a}$ is *robust to coalitional deviations* if there is no nonempty coalition $M \subseteq N$ and no other action profile $\mathbf{a}'$ so that:

(i) $a'_i = 0$ for all $i \notin M$;
(ii) each $i \in M$ weakly prefers $\mathbf{a}'$ to $\mathbf{a}$;
(iii) some $i \in M$ strictly prefers $\mathbf{a}'$ to $\mathbf{a}$.

Action profiles robust to coalitional deviations correspond to those that are in the β-core, which in this environment are the same as those in the α-core. The α-core is defined by a deviating coalition first choosing their actions to maximize their payoffs and then the other players choosing actions to punish the deviating coalition given what has happened. The β-core is defined by the non-deviating players first choosing their actions to punish the deviating coalition, and then the deviators choosing actions given that (Aumann and Peleg, 1960). In our setting, as action levels of 0 for the non-deviating players always minimize the payoffs of each member of a deviating coalition, the order of the moves does not matter.

We now state and prove a formal version of the claim made in Section 4.2.3.

Proposition **OA8.** If $\mathbf{a} \in \mathbb{R}^n_+$ is a centrality action profile, then $\mathbf{a}$ is robust to coalitional deviations.

Proof of Proposition OA8: Applying Theorem 1, we will work with the Lindahl outcomes rather than the centrality action profiles. Let $\mathbf{a}^* \in \mathbb{R}^n_+$ be a Lindahl outcome and $\mathbf{P}$ the associated price matrix, satisfying the conditions of Definition 5 (recall that this is an equivalent definition of a Lindahl outcome, given in the proof of Theorem 1 above). Then we have

$$\mathbf{a}^* \in \operatorname{argmax} u_i(\mathbf{a}) \text{ s.t. } \mathbf{a} \in \mathbb{R}^n_+ \text{ and } \sum_{j\in N} P_{ij} a_j \leq 0.$$

[24]In particular, we see $H \in \mathcal{F}$.

We will refer to this convex program as the Lindahl problem. We now use these properties of $\mathbf{a}^*$ to show that it is robust to coalitional deviations. Pareto efficiency of $\mathbf{a}^*$, which follows by Proposition 1, ensures the grand coalition doesn't have a profitable deviation. We now rule out all other possible coalitional deviations. Toward a contradiction, suppose $\mathbf{a}^*$ is not robust to coalitional deviations, and therefore that there exists a nonempty proper coalition M and an $\mathbf{a}'$ (with $a'_i = 0$ for $i \notin M$) for which $u_i(\mathbf{a}') \geq u_i(\mathbf{a}^*)$ for each $i \in M$, with strict inequality for some $i \in M$. Since $\mathbf{a}^*$ solves the Lindahl problem, we must have that the action profile $\mathbf{a}'$ is weakly unaffordable to i at prices $\mathbf{P}$: $\sum_{j \in N} P_{ij} a'_j \geq 0$ for each $i \in M$.[25]

There are then two cases to consider. Suppose first that there is some $i \in M$ such that $u_i(\mathbf{a}') > u_i(\mathbf{a}^*)$ and so $\sum_{j \in N} P_{ij} a'_j > 0$. If this is true, then:

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j > 0. \tag{OA-11}$$

On the other hand,

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j = \sum_{j \in M} a'_j \sum_{i \in M} P_{ij} \leq \sum_{j \in M} a'_j \sum_{i \in N} P_{ij} = 0. \tag{OA-12}$$

The first equality follows by switching the order of summation, the inequality holds because $P_{ij} \geq 0$ for $j \neq i$, and the final equality follows from $P_{ii} = -\sum_{j: j \neq i} P_{ji}$ for all i. Equation (OA-12) contradicts equation (OA-11). ∎

OA6. Irreducibility of the Benefits Matrix

In Assumption 3, we posited that $\mathbf{B}(\mathbf{a})$ is irreducible—i.e., that it is not possible to find an outcome and a partition of society into two nonempty groups such that, at that outcome, one group does not care about the effort of the other at the margin.

How restrictive is this assumption? We now discuss how our analysis extends beyond it. Suppose that whether $B_{ij}(\mathbf{a})$ is positive or 0 does not depend on $\mathbf{a}$, so that the directed graph describing whose effort matters to whom is constant, though the nonzero marginal benefits may change as we vary $\mathbf{a}$. Let $\mathbf{G}$ be a matrix defined by

$$G_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } B_{ij}(\mathbf{a}) > 0 \text{ for all } \mathbf{a} \\ 0 & \text{otherwise.} \end{cases}$$

We say a subset $S \subseteq N$ is *closed* if $G_{ij} = 0$ for every $i \in S$ and $j \notin S$. We say S is *irreducible*[26] if $\mathbf{G}$ is irreducible when restricted to S.

We can always partition N into some closed, irreducible subsets

$$S^{(1)}, S^{(2)}, \ldots, S^{(m)}$$

and a remaining class T of agents who are in no closed, irreducible subset. The utility of any agent in a set $S^{(k)}$ is independent of the choices of anyone outside the set (and these are the minimal sets with that property). So it seems reasonable to consider

[25] Suppose $\sum_{j \in N} P_{ij} a'_j < 0$ for some $i \in M$. It follows that, while satisfying the assumption $\sum_{j \in N} P_{ij} a'_j \leq 0$, every a_j for $j \neq i$ can be increased slightly; by Assumption 3, this makes i better off.

[26] For more details on how the Perron-Frobenius theory extends to the non-irreducible case, see (Meyer, 2000, pp. 694–695).

negotiations restricted to each such set; that is, to take the set of players to be $S^{(k)}$. All our analysis then goes through without modification on each such subset.

When entries $B_{ij}(\mathbf{a})$ change from positive to zero depending on $\mathbf{a}$, then the analysis becomes substantially more complicated, and we leave it for future work.

OA7. Endogenous Status Quo: Comparing Nash and Lindahl in a Star Example

This section builds on the example in Section 5.2 which analyzed regular graphs. To shed more light on how Nash and efficient outcomes are related, this section considers a particular asymmetric graph. As before, $\mathbf{G}$ is an undirected, unweighted graph ($g_{ij} = g_{ji} \in \{0,1\}$), with no self-links ($g_{ii} = 0$) describing which agents are neighbors. However, now we let $\mathbf{G}$ have a star structure with 4 agents in which agent 1 is the center agent, linked to all other agents, and there are no other links. Suppose utility functions have the following functional form

$$u_i(\mathbf{a}) = \log\left(a_i + \delta \sum_j g_{ij} a_j\right) - a_i.$$

For $\delta \in [0, 1/3)$, results from Ballester, Calvó-Armengol and Zenou (2006) and Bramoullé, Kranton, and d'Amours (2014) imply that there is a unique interior Nash equilibrium in which $a_1^{\mathrm{NE}} = (1-3\delta)/(1-3\delta^2)$, and $a_j^{\mathrm{NE}} = (1-\delta)/(1-3\delta^2)$ for $j \neq 1$.

We will compare the Lindahl equilibrium actions to the Nash actions. We begin by arguing that in any Lindahl outcome, all periphery agents $i = 2, 3, 4$ take the same action. First, because the Lindahl outcome is Pareto efficient,[27] by Lemma 1, it follows that Lindahl actions are strictly higher than Nash actions. As in the main text, take the Nash equilibrium as the status quo; denote the increments over it $\widehat{a}_i$; and denote the corresponding utility functions by $\widehat{u}_i$. As we have just argued, in the Lindahl outcome each agent takes some action $\widehat{a}_i > 0$. By the extension of Theorem 1, the periphery agents take actions

$$\widehat{a}_i = \frac{\partial \widehat{u}_i / \partial \widehat{a}_1}{-\partial \widehat{u}_i / \partial \widehat{a}_i} \widehat{a}_1 \quad \text{ or } f_i(\widehat{a}_i) := \quad \widehat{a}_i \frac{-\partial \widehat{u}_i / \partial \widehat{a}_i}{\partial \widehat{u}_i / \partial \widehat{a}_1} = \widehat{a}_1.$$

Given the functional form utilities take, f_i is monotonic at positive arguments. Thus, fixing the action the center agent agent takes such that $\widehat{a}_1 > 0$, the action of each other agent $i \neq 1$ in a Lindahl outcome is the same, and uniquely determined.

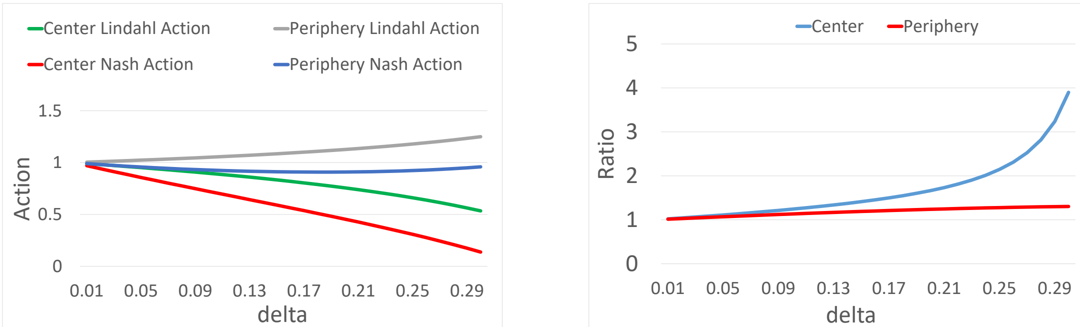
So the benefits matrix has the following form:

$$\mathbf{B}(\mathbf{a}) = \begin{pmatrix} 0 & \delta/(a_1 + 3\delta a_2 - 1) & \delta/(a_1 + 3\delta a_2 - 1) & \delta/(a_1 + 3\delta a_2 - 1) \\ \delta/(a_2 + \delta a_1 - 1) & 0 & 0 & 0 \\ \delta/(a_2 + \delta a_1 - 1) & 0 & 0 & 0 \\ \delta/(a_2 + \delta a_1 - 1) & 0 & 0 & 0 \end{pmatrix}.$$

The extension of Theorem 1 now implies that, for some scalar $s > 0$, we have

$$\left(s \frac{\sqrt{3\delta/(a_1^{\mathrm{LE}} + 3\delta a_2^{\mathrm{LE}} - 1)}}{\sqrt{\delta/(a_2^{\mathrm{LE}} + \delta a_1^{\mathrm{LE}} - 1)}}, s, s, s \right) = (\mathbf{a}^{\mathrm{LE}} - \mathbf{a}^{\mathrm{NE}}). \tag{OA-13}$$

[27] By Foley (1970) or the argument we gave after Theorem 1, but replacing Theorem 1 by its extension to the Nash status quo case.



(A) Lindahl and Nash Actions

(B) Ratio of Lindahl to Nash Actions

FIGURE 1. Panel (a) shows the Nash equilibrium action levels and Lindahl action levels for the periphery and center players in a 4 person star network as externalities increase. Panel (b) shows the ratio of the Lindahl actions to the Nash equilibrium actions for the center and periphery players as externalities increase.

Moreover, the spectral radius of $\mathbf{B}(\mathbf{a})$ being 1 corresponds to the equation

$$3\sqrt{3\delta/(a_1^{\mathrm{LE}} + 3\delta a_2^{\mathrm{LE}} - 1)}\sqrt{\delta/(a_2^{\mathrm{LE}} + \delta a_1^{\mathrm{LE}} - 1)} = 1.$$

Taking this equation along with the first two equations in the system (OA-13) yields a system of 3 (distinct) equations with three unknowns. Solving this system for varying values of δ, the Lindahl equilibrium actions are plotted in Figure 1.

Figure 1 shows that as the strength of the positive externalities increases the center agent's Nash equilibrium action decreases. Intuitively, as externalities increase the center agent is able to free ride more on the actions of the other agents and reduces her own action.[28] Although the Lindahl actions are Pareto efficient and internalize the externalities through market forces, a similar pattern is observed for the Lindahl actions. As externalities increase, the center agent's Lindahl action decreases. Holding the benefits received from periphery agents constant, increasing externalities mean that the center agent benefits more at the margin from the actions of the periphery agents. However, holding the actions of the periphery agents constant, the center agent receives higher benefits, and as these benefits increase diminishing marginal utility sets in. As periphery agents consume less, the effect of diminishing marginal utility is less pronounced for them. Ultimately this results in the center agent taking a lower action in the Lindahl equilibrium as externalities increase, while the periphery agents take higher actions. These comparative statics contrast with the analysis of regular graphs in Section 5.2. In regular graphs, for the functional form of utilities used in this section, agents' Lindahl actions are invariant with respect to externalities. Nevertheless, as in Section 5.2, the affect of the externalities being internalized

[28]The periphery agents have less scope for free-riding as they are connected to just one other agent. At first the free-riding effect leads them to take lower Nash actions as externalities increase, but after a while the reduced action of the center agent dominates and they take higher actions.

through market forces in the Lindahl equilibrium is visible when comparing the Lindahl actions to the Nash actions. As shown in Panel (b), the ratio of Lindahl to Nash actions increases as externalities increase. This increase is particularly pronounced for the center agent.

OA8. Explicit Formulas for Lindahl Outcomes

OA8.1. **A Parametric Family of Preferences and a Formula for Centrality Action Profiles.** Here we provide more interpretations regarding explicit formulas for Lindahl outcomes, following up on the discussion of Section 5.3. In that section, we defined:[29]

$$u_i(\mathbf{a}) = -a_i + \sum_j \left[G_{ij} a_j + H_{ij} \log a_j\right].$$

for non-negative matrices $\mathbf{G}$ and $\mathbf{H}$ with zeros on the diagonal, assuming $r(\mathbf{G}) < 1$. Letting $h_i = \sum_j H_{ij}$, the (eigenvector) centrality property of actions boils down to $\mathbf{a} = \mathbf{h} + \mathbf{G}\mathbf{a}$ or

$$\mathbf{a} = (\mathbf{I} - \mathbf{G})^{-1}\mathbf{h}. \tag{OA-14}$$

Note that the vector $\mathbf{a}$ is well-defined and nonnegative[30] by the assumption that $r(\mathbf{G}) < 1$. These centrality action profiles (in the sense used throughout our paper) correspond to agents' degree centralities, Bonacich centralities, or eigenvector centralities on some network $\mathbf{M}$, for specific parametrizations of the above utility functions.

OA8.2. **Degree Centrality.** To obtain agents' degree centralities as their centrality actions, we set $\mathbf{H} = \mathbf{M}$ and let $\mathbf{G} = \mathbf{0}$. Then equation (OA-14) says that $\mathbf{a} = \mathbf{h}$. When costs are linear in one's own action and benefits are logarithmic in others' actions, then an agent i's contribution is determined by how much he benefits from everyone else's effort at the margin: the sum of coefficients H_{ij} as j ranges across the other agents. The agents who are particularly dependent on the rest are the ones who are contributing the most.

OA8.3. **Bonacich Centrality.** To obtain agents' Bonacich centralities as their centrality actions, we set $\mathbf{G} = \alpha\mathbf{M}$ for $\alpha < 1/r(\mathbf{M})$, and let each row of $\mathbf{H}$ sum to 1. Dropping the arguments, the defining equation for Bonacich centrality[31] says that for every i, we have:

$$\beta_i = 1 + \alpha \sum_j M_{ij}\beta_j.$$

Thus, every node gets a baseline level of centrality (one unit) and then additional centrality in proportion to the centrality of those it is linked to. To shed further light on this result, recall the definitions and notation related to walks from Section 5, and let

[29] These should be viewed as functions $u_i : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$, with $0 \cdot \log 0$ understood as 0. In other words, preferences should be completed by continuity to the extended range. No result in the paper is affected by this slight departure from the framework of Section 2.

[30] See Ballester, Calvó-Armengol, and Zenou (2006, Section 3).

[31] An important antecedent was discussed by Katz (1953).

$$V_i(\ell;\mathbf{M}) = \sum_{w \in \mathcal{W}_i^{\downarrow}(\ell;\mathbf{M})} v(w;\mathbf{M}).$$

This is the sum of the values of all walks of length ℓ in $\mathbf{M}$ ending at i. Then we have:

FACT **OA1.** $\beta_i(\mathbf{M},\alpha) = 1 + \sum_{\ell=1}^{\infty} \alpha^{\ell} V_i(\ell;\mathbf{M}^{\mathsf{T}})$.

Fact OA1 is established, e.g., in Ballester, Calvó-Armengol, and Zenou (2006, Section 3). Thus, the Bonacich centrality is equal to 1 plus a weighted sum of values of all walks in $\mathbf{M}^{\mathsf{T}}$ terminating at i, with longer walks downweighted exponentially.

In contrast to the case of degree centrality treated in the previous section, it is not only how much i benefits from his immediate neighborhood that matters in determining his contribution, but also how much i's neighbors benefit from *their* neighbors, etc.

OA8.4. **Eigenvector Centrality.** Eigenvector centrality is a key notion throughout the paper. Theorem 1 establishes a general connection between eigenvector centrality and Lindahl outcomes. However, this theorem characterizes $\mathbf{a}$ through an *endogenous* eigenvector centrality condition—a condition that depends on $\mathbf{B}(\mathbf{a})$. In this section, we study the special case in which action levels approximate eigenvector centralities defined according to an *exogenous* network.

We continue with the specification from Section OA8.3, with one exception: We consider networks $\mathbf{M}$ such that $r(\mathbf{M}) = 1$.[32] Thus,

$$\mathbf{a} = \boldsymbol{\beta}(\mathbf{M},\alpha).$$

By the Perron–Frobenius Theorem, $\mathbf{M}$ has a unique right-hand Perron eigenvector $\mathbf{e}$ (satisfying $\mathbf{e} = \mathbf{M}\mathbf{e}$) with entries summing to 1. As we take the limit $\alpha \to 1$, agents' Bonacich centralities become large but $a_i/a_j \to e_i/e_j$, for every i, j. That is, each agent's share of the total of all actions converges to his eigenvector centrality according to $\mathbf{M}$.[33] The reason for this convergence is presented in the proof of Theorem 3 of Golub and Lever (2010); see also Bonacich (1991).

[32]This is just a normalization here: For any $\mathbf{M}$, we can work with the matrix $(1/r(\mathbf{M}))\mathbf{M}$, as this has spectral radius 1.

[33]To loosely gain some intuition for this, note that $\mathbf{a} = \alpha\mathbf{M}\mathbf{a} + \mathbf{1}$; as $\alpha \to 1$, actions grow large and we can think of this equation as saying $\mathbf{a} \approx \mathbf{M}\mathbf{a}$.

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