LEARNING FROM NEIGHBORS ABOUT A CHANGING STATE ONLINE APPENDIX

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This document contains supporting material for the paper Dasaratha, Golub, and Hak (2023): "Learning from Neighbors about a Changing State," which herein we refer to as the "main paper" or simply "paper."

OA1. Numerical results in real networks

The message of Section 4 is that signal diversity enables good aggregation, and signal homogeneity obstructs it. The theoretical results in that section, however, were asymptotic, and the good-aggregation result used some assumptions on the distribution of graphs. In this section we show that the substantive message applies to realistic networks with moderate degrees. We do this by computing equilibria for actual social networks from the data in Banerjee, Chandrasekhar, Duflo, and Jackson (2013a). This data set contains the social networks of villages in rural India.[1] There are 43 networks in the data, with an average network size of 212 nodes (standard deviation = 53.5), and an average degree of 19 (standard deviation = 7.5).

Our simulation exercises measure the benefits of heterogeneity for equilibrium aggregation. For each network, we calculate the equilibrium with $\rho = 0.9$ for two types of environments. The first is the homogeneous case, with all signal variances set to 2. The second is a heterogeneous case, where half of the agents have a signal variance greater than 2 and half of villagers have a signal variance less than 2, chosen to hold constant the total amount of information that reaches the community via private signals. That is, we set the signal variances so that the average precision in each village is $\frac{1}{2}$, as in the homogeneous case. This signal assignment holds fixed the average utility when all villagers are autarkic, or equivalently holds fixed the average utility when all villagers know the state θ_{t-1} in the previous period exactly. At the same time, it varies the level of heterogeneity in signal endowments. Villagers are randomly assigned to better or worse private signals, and the simulation results do not depend substantially on the realized random assignment. Our outcomes will be the average social signal error variance in each village and the average social signal error variance across all villages.

It is useful to begin by looking at the equilibrium average aggregation errors, i.e., social signal variances, in the case of homogeneous signals. This is the horizontal coordinate in Figure OA1.1(a); each village is a data point, and the points have a standard deviation of 0.013. In this case, differences in learning outcomes are due only to differences in the network structure, and we will call this number the *network-driven variation*. Now we introduce some private signal diversity. In

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[1] We take the networks that were used in the estimation in Banerjee, Chandrasekhar, Duflo, and Jackson (2013b). As in their work, we take every reported relationship to be reciprocal for the purposes of sharing information. This makes the graphs undirected.

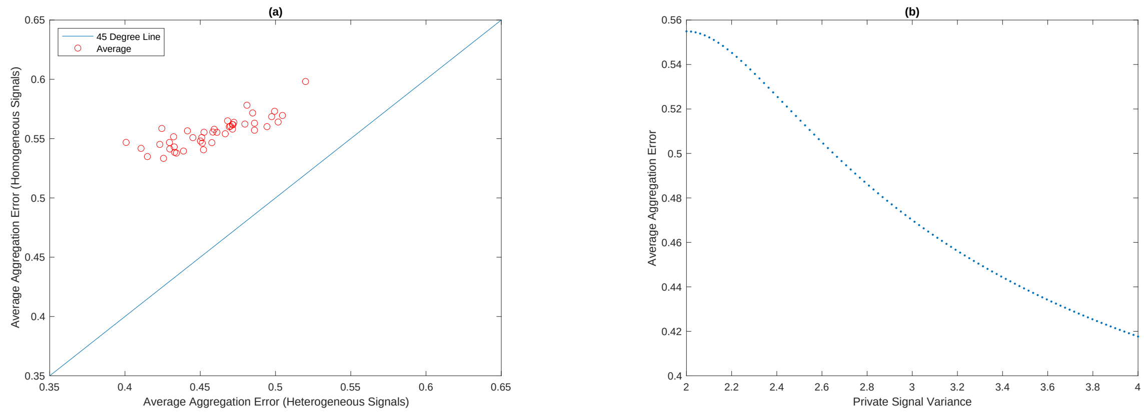

FIGURE OA1.1. Social signal variance in Indian villages. (a) The average social signal variance of agents in each village, in the homogeneous and heterogeneous cases. In the homogeneous case all agents have private signal variance 2. In the heterogeneous case, half of agents have private signal variance $\frac{3}{2}$ and half of agents have private signal variance 3. (b) The average social signal variance for all agents as we vary the worse private signal variance from 2 to 4 and hold fixed the average precision of private signals.

our first exercise, we change the variance of the worse private signal from 2 (homogeneous signals) to 3 (heterogeneous signals), and adjust the other variance as discussed above to hold fixed the total amount of information coming into the network. The vertical coordinate in Figure OA1.1(b) depicts the equilibrium aggregation error in each village. The average of this number across all villages falls to 0.470, compared to 0.555 (in the homogeneous case). Therefore, adding heterogeneity by increasing the private signal variance for half of the agents by 50% changes social signal error variance by 6.5 times the network-driven variation. Learning is much better with some private signal heterogeneity than in villages with very favorable networks (i.e., those that achieve the best aggregation under homogeneous signals).

In Figure OA1.1(b), rather than working with the particular choice of 3 for the variance of the private signal, we look across all choices of this variance between 2 and 4 and plot the average equilibrium social signal variance across all villages.

Figure OA1.1(b) also sheds light on the value of a small amount of heterogeneity. The results in Section 4 can be summarized as saying that, to achieve the aggregation benchmark of essentially knowing the previous period's state, there need to be at least two different private signal variances in the network. Formally, this is a knife-edge result: As long as private signal variances differ at all, then as $n \to \infty$, aggregation errors vanish; with exactly homogeneous signal endowments, aggregation errors are much higher. The figure shows that the transition from the first regime to the second is actually gradual. In particular, a very small amount of heterogeneity provides little benefit in finite networks, as there is not enough diversity of signal endowments for villagers to anti-imitate. However, a 50% change in the variance of one of the signals (equivalently, a 22% change in its standard deviation) makes the community much better able to use the same total amount of information.

OA2. Identification and testable implications

One of the main advantages of the parametrization we have studied is that standard methods can easily be applied to estimate the model and test hypotheses within it. The key feature making the model econometrically well-behaved is that, in the solutions we focus on, agents' actions are linear functions of the random variables they observe. Moreover, the evolution of the state and arrival of information creates exogenous variation. We briefly sketch how these features can be used for estimation and testing.

Assume the following. The analyst obtains noisy measurements $\overline{a}_{i,t} = a_{i,t} + \xi_{i,t}$ of agent's actions (where $\xi_{i,t}$ are i.i.d., mean-zero error terms). He knows the parameter ρ governing the stochastic process, but may not know the network structure or the qualities of private signals $(\sigma_i)_{i=1}^n$. Suppose also that the analyst observes the state θ_t ex post (perhaps with a long delay).[2] For example, agents may be trying to forecast the current state of an economic variable such as GDP or employment, which is later measured.

Now, consider *any* steady state in which agents put constant weights W_{ij} on their neighbors and w_i^s on their private signals over time. We will discuss the case of $m = 1$ to save on notation, though all the statements here generalize readily to arbitrary m.

We first consider how to estimate the weights agents are using, and to back out the structural parameters of our model when it applies. The strategy does not rely on uniqueness of equilibrium. We can identify the weights agents are using through standard vector autoregression methods. In steady state,

$$\overline{a}_{i,t} = \sum_j W_{ij}\rho\overline{a}_{j,t-1} + w_i^s\theta_t + \zeta_{i,t}, \tag{OA-1}$$

where $\zeta_{i,t} = w_i^s\eta_{i,t} - \sum_j W_{ij}\rho\xi_{j,t-1} + \xi_{i,t}$ are error terms i.i.d. across time. The first term of this expression for $\zeta_{i,t}$ is the error of the signal that agent i receives at time t. The summation combines the measurement errors from the observations $\overline{a}_{j,t-1}$ from the previous period.[3] Thus, we can obtain consistent estimators $\widetilde{W}_{ij}$ and $\widetilde{w}_i^s$ for W_{ij} and w_i^s, respectively.

We now turn to the case in which agents are using *equilibrium* weights. First, and most simply, our estimates of agents' equilibrium weights allow us to recover the network structure. If the weight $\widehat{W}_{ij}$ is nonzero for any i and j, then agent i observes agent j. Generically the converse is true: if i observes j then the weight $\widehat{W}_{ij}$ is nonzero. Thus, network links can generically be identified by testing whether the recovered social weights are nonzero. For such tests (and more generally) the standard errors in the estimators can be obtained by standard techniques.[4]

Now we examine the more interesting question of how structural parameters can be identified assuming an equilibrium is played, and also how to test the assumption of equilibrium.

The first step is to compute the empirical covariances of action errors from observed data; we call these $\widetilde{V}_{ij}$. Under the assumption of equilibrium, we now show how to determine the signal variances using the fact that equilibrium is characterized by $\Phi(\widehat{\boldsymbol{V}}) = \widehat{\boldsymbol{V}}$ and recalling the explicit

[2] We can instead assume that the analyst observes (a proxy for) the private signal $s_{i,t}$ of agent i; we mention how below.

[3] This system defines a VAR(1) process (or generally VAR(m) for memory length m).

[4] Methods involving regularization may be practically useful in identifying links in the network. Manresa (2013) proposes a regularization (LASSO) technique for identifying such links (peer effects). In a dynamic setting such as ours, with serial correlation, the techniques required will generally be more complicated.

formula (3.3) for Φ. In view of this formula, the signal variances σ_i^2 are uniquely determined by the other variables:

$$\widehat{V}_{ii} = \sum_j \sum_k \widehat{W}_{ij} \widehat{W}_{ik} \left(\rho^2 \widehat{V}_{jk} + 1\right) + (\widehat{w}_i^s)^2 \sigma_i^2. \tag{OA-2}$$

Replacing the model parameters other than σ_i^2 by their empirical analogues, we obtain a consistent estimate $\widetilde{\sigma}_i^2$ of σ_i. This estimate could be directly useful—for example, to an analyst who wants to identify the best-informed "expert" from the network and ask about her private signals.

Note that our basic VAR for recovering the weights relies only on constant linear strategies and does not assume that agents are playing any particular strategy within this class. Thus, if agents are using some other behavioral rule (e.g., optimizing in a misspecified model) we can replace (OA-2) by a suitable analogue that reflects the bounded rationality in agents' inference. If such a steady state exists, and using the results in this section, one can create a statistical test of how agents are behaving. For instance, we can test the hypothesis that they are Bayesian against the naive alternative of our Section 5.1.

OA3. Details of definitions

OA3.1. **Exogenous random variables.** Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\nu_t, \eta_{i,t})_{t \in \mathbb{Z}, i \in N}$ be normal, mutually independent random variables, with ν_t having variance 1 and $\eta_{i,t}$ having variance σ_i^2. Also take a stochastic process $(\theta_t)_{t \in \mathbb{Z}}$, such that for each $t \in \mathbb{Z}$, we have (for $0 < |\rho| \le 1$)

$$\theta_t = \rho \theta_{t-1} + \nu_t.$$

Such a stochastic process exists by standard constructions of the AR(1) process or, in the case of $\rho = 1$, of the Gaussian random walk on a doubly infinite time domain. Define $s_{i,t} = \theta_t + \eta_{i,t}$.

OA3.2. **Formal definition of game and stationary linear equilibria.** We now fill in the details of our Bayesian game.

Players and strategies. The set of players (or agents) is $\mathcal{A} = \{(i,t) : i \in N, t \in \mathbb{Z}\}$. The set of (pure) *responses* of an agent (i,t) is defined to be the set of all Borel-measurable functions $\xi_{(i,t)} : \mathbb{R} \times (\mathbb{R}^{|N(i)|})^m \to \mathbb{R}$, mapping her own signal and her neighborhood's actions, $\left(s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m\right)$, to a real-valued action $a_{i,t}$. We call the set of these functions $\widetilde{\Xi}_{(i,t)}$. Let $\widetilde{\Xi} = \prod_{(i,t) \in \mathcal{A}} \widetilde{\Xi}_{(i,t)}$ be the set of response profiles. We now define the set of *(unambiguous) strategy profiles*, $\Xi \subset \widetilde{\Xi}$. We say that a response profile $\xi \in \widetilde{\Xi}$ is a strategy profile if the following two conditions hold

1. There is a tuple of real-valued random variables $(a_{i,t})_{i \in N, t \in \mathbb{Z}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $(i,t) \in \mathcal{A}$, we have
$$a_{i,t} = \xi_{(i,t)} \left(s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m\right).$$
2. Any two tuples of real-valued random variables $(a_{i,t})_{i \in N, t \in \mathbb{Z}}$ satisfying Condition 1 are equal almost surely.

That is, a response profile is a strategy profile if there is an essentially unique specification of behavior that is consistent with the responses: i.e., if the responses uniquely determine the behavior of the population, and hence payoffs.[5] Note that if $\xi \in \Xi$, then it can be checked that $\widetilde{\xi} = (\xi'_{(i,t)}, \xi_{-(i,t)}) \in \Xi$

[5]Condition 1 is necessary to rule out response profiles such as the one given by $\xi_{i,t}\left(s_{i,t}, a_{i,t-1}\right) = |a_{i,t-1}| + 1$. This profile, despite consisting of well-behaved functions, does not correspond to any specification of behavior for the whole

whenever $\xi'_{(i,t)} \in \widetilde{\Xi}_{(i,t)}$. Thus, if we start with a strategy profile and consider agent (i,t)'s deviations, they are unrestricted: she may consider any response.

Payoffs. The payoff of an agent (i,t) under any strategy profile $\xi \in \Xi$ is

$$u_{i,t}(\xi) = -\mathbb{E}\left[(a_{i,t} - \theta_t)^2\right] \in [-\infty, 0],$$

where the actions $a_{i,t}$ are taken according to $\xi_{(i,t)}$ and the expectation is taken in the probability space we have described. This expectation is well-defined because inside the expectation there is a nonnegative, measurable random variable, for which an expectation is always defined, though it may be infinite.

Equilibria. A (Nash) *equilibrium* is defined to be a strategy profile $\xi \in \Xi$ such that, for each $(i,t) \in \mathcal{A}$ and each $\widetilde{\xi} \in \Xi$ such that $\widetilde{\xi} = (\xi'_{(i,t)}, \xi_{-(i,t)})$ for some $\xi'_{(i,t)} \in \Xi_{(i,t)}$, we have

$$u_{i,t}(\widetilde{\xi}) \leq u_{i,t}(\xi).$$

For $p \in \mathbb{Z}$, we define the shift operator $\mathfrak{T}_p$ to translate variables to time indices shifted p steps forward. This definition may be applied, for example, to Ξ.[6] A strategy profile $\xi \in \Xi$ is *stationary* if, for all $p \in \mathbb{Z}$, we have $\mathfrak{T}_p \xi = \xi$.

We say $\xi \in \Xi$ is a *linear* strategy profile if each ξ_i is a linear function. Our analysis focuses on *stationary, linear equilibria.*

OA4. Remaining proofs

OA4.1. **Details of calculations for equation (3.1) for best-response actions.**

Let $P = \mathcal{N}(\mu_{\text{prior}}, \sigma^2_{\text{prior}})$ be a normal prior over θ_t.

First, we will establish that for any $\ell \geq 0$ we have

$$\mathbb{E}_P[\theta_t \mid \rho^\ell a_{i,t-\ell}] = \rho^\ell a_{i,t-\ell}. \tag{OA-1}$$

Note that

$$\theta_t = \rho^\ell \theta_{t-\ell} + \sum_{k=1}^{\ell} \rho^{\ell-k} \nu_{t-\ell+k}.$$

Take conditional expectations $\mathbb{E}_P[\,\cdot \mid \rho^\ell a_{i,t-\ell}]$ on both sides and note that the summation makes no contribution, since the innovations $\nu_{t-\ell+k}$ are independent of $a_{i,t-\ell}$ and have mean zero. Now we will show that $\mathbb{E}_P[\theta_{t-\ell} \mid a_{i,t-\ell}] = a_{i,t-\ell}$:

$$\begin{aligned}
\mathbb{E}_P[\theta_{t-\ell} \mid a_{i,t-\ell}] &= \mathbb{E}_P[\mathbb{E}_P[\theta_{t-\ell} \mid z_{i,t-\ell}] \mid a_{i,t-\ell}] && \text{tower property} \\
&= \mathbb{E}_P[a_{i,t-\ell} \mid a_{i,t-\ell}] && \text{eq. (2.3) in the paper.}
\end{aligned}$$

This completes the proof of (OA-1).

Now we turn to the updating formula (3.1). We will transform the updating problem into a canonical form to apply standard results on Bayesian conditioning with normal distributions.[7] Since the updating formula is important, we will spell out most of the details.

population (because time extends infinitely backward). Condition 2 is necessary to rule out response profiles such has the one given by $\xi_{i,t}\left(s_{i,t}, a_{i,t-1}\right) = a_{i,t-1}$, which have many satisfying action paths, leaving payoffs undetermined.

[6] I.e., $\sigma' = \mathfrak{T}_p \sigma$ is defined by $\sigma_{(i,t)} = \sigma_{(i,t-p)}$.

[7] This turns out to be a bit involved because, under a proper prior, we cannot write $z_{i,t}$ as $\theta_t \mathbf{1}$ plus noise independent of θ_t, which is the case where the formulas for $\mathbb{E}[\theta_t \mid z_{i,t}]$ would be most straightforward.

Fix an agent (i, t) and write $\boldsymbol{z} := \boldsymbol{z}_{i,t}$. Define the (random) vector $\boldsymbol{v}$ to consist of the time-t state θ_t and the recent innovations $\nu_{t'}$ for $t' \in \{t, t-1, \ldots, t-m+1\}$. The agent's information $\boldsymbol{z}$ can be written as $\boldsymbol{z} = \boldsymbol{H}\boldsymbol{v} + \boldsymbol{w}$ where $\boldsymbol{H}$ is a known matrix (depending on parameters of the environment) and $\boldsymbol{w}$ is a vector of Gaussian errors, independent of $\boldsymbol{v}$ and of one another. Then standard results on Gaussian updating (Kay, 1993, Theorem 10.3) imply that, fixing some prior P over θ_t, the posterior mean of θ_t can be expressed as

$$Y := \mathbb{E}_P[\theta_t \mid \boldsymbol{z}] = \beta\mu_{\text{prior}} + (1-\beta)\boldsymbol{\lambda}^*\boldsymbol{z}, \tag{OA-2}$$

where μ_{prior} is the prior mean of θ_t, the vector $\boldsymbol{\lambda}^*$ sums to 1, and $\beta \in \mathbb{R}$.[8]

We will now give a self-contained calculation of $\boldsymbol{\lambda}^*$. A conditional expectation Y minimizes $\mathbb{E}_P[(Y-\theta_t)^2]$ among measurable functions of $\boldsymbol{z}$, so, fixing β, the vector $\boldsymbol{\lambda}^*$ must minimize the error $\mathbb{E}[(\boldsymbol{\lambda}\boldsymbol{z}-\theta_t)^2]$ among $\boldsymbol{\lambda}$ summing to 1. We use this fact to characterize $\boldsymbol{\lambda}^*$. Note that the following equations hold whenever $\boldsymbol{\lambda}\mathbf{1} = 1$.

$$\begin{aligned}
\mathbb{E}[(\boldsymbol{\lambda}\boldsymbol{z}-\theta_t)^2] &= \operatorname{Var}\left[\sum_k \lambda_k(z_k-\theta_t)\right] \\
&= \sum_{k,k'} \lambda_k\lambda_{k'} \operatorname{Cov}\left[z_k-\theta_t, z_{k'}-\theta_t\right] \\
&= \boldsymbol{\lambda}\boldsymbol{C}\boldsymbol{\lambda}^\top,
\end{aligned}$$

where $\boldsymbol{C}$ is the covariance matrix on the second line (denoted by $\boldsymbol{C}_{i,t-1}$ in Section 3.1 of the paper). Thus $\boldsymbol{\lambda}^*$ may be characterized as minimizing $\boldsymbol{\lambda}\boldsymbol{C}\boldsymbol{\lambda}^\top$ subject to $\boldsymbol{\lambda}\mathbf{1} = 1$. The first-order conditions give that $\boldsymbol{\lambda}^*\boldsymbol{C} = \gamma\mathbf{1}$ for some $\gamma \in \mathbb{R}$, and then using the constraint gives the solution

$$\boldsymbol{\lambda}^* = \frac{\mathbf{1}^\top\boldsymbol{C}^{-1}}{\mathbf{1}^\top\boldsymbol{C}^{-1}\mathbf{1}}.$$

Note this does not depend on the prior P.

Finally, by standard results on Bayes' rule with Gaussian distributions, as the prior becomes diffuse (i.e., the precision $\sigma_{\text{prior}}^{-2}$ tends to 0) we have $\beta \to 0$.[9]

Putting everything together, we have that in the diffuse-prior limit,

$$\mathbb{E}_P[\theta_t \mid \boldsymbol{z}] \to \frac{\mathbf{1}^\top\boldsymbol{C}^{-1}}{\mathbf{1}^\top\boldsymbol{C}^{-1}\mathbf{1}}\boldsymbol{z}.$$

OA4.2. **Proof of Proposition 2.** We first check there is a unique equilibrium and then prove the remainder of Proposition 2.

Lemma OA1. Suppose G has symmetric neighbors. Then there is a unique equilibrium.

[8]The cited result states $\mathbb{E}_P[\theta_t \mid \boldsymbol{z}] = \beta\mu_{\text{prior}} + \gamma\boldsymbol{\lambda}^*\boldsymbol{z}$ for a $\boldsymbol{\lambda}^*$ we can normalize to sum to 1, and some $\beta, \gamma \in \mathbb{R}$. Take unconditional expectations on both sides to obtain $\mu_{\text{prior}} = \beta\mu_{\text{prior}} + \gamma\mu_{\text{prior}}$, using $\sum_k \lambda_k = 1$ and $\mathbb{E}_P[z_k] = \mu_{\text{prior}}$ for every k. (For the last assertion, by (OA-1) and the law of iterated expectations, we have $\mathbb{E}_P[z_k] = \mathbb{E}_P[\mathbb{E}_P[\theta_t \mid z_k]] = \mathbb{E}_P[\theta_t]$.) This can hold for all priors P only if $\gamma = 1-\beta$.

[9]The vector $(\theta_t, \boldsymbol{\lambda}^*\boldsymbol{z})$ is jointly Gaussian, and in the diffuse-prior limit the error $\boldsymbol{\lambda}^*\boldsymbol{z} - \theta_t$ has mean zero and is uncorrelated with θ_t. Thus, in that limit, the formula (Kay, 1993, Theorem 10.2) for $\mathbb{E}_P[\theta_t \mid \boldsymbol{\lambda}^*\boldsymbol{z}]$ converges to an average of μ_{prior} and $\boldsymbol{\lambda}^*\boldsymbol{z}$, weighted in proportion to their precisions. Finally, we have $\mathbb{E}_P[\theta_t \mid \boldsymbol{z}] = \mathbb{E}_P[\theta_t \mid \boldsymbol{\lambda}^*\boldsymbol{z}]$ by (OA-2).

Proof of Lemma OA1. We will show that when the network satisfies the condition in the proposition statement, Φ induces a contraction on a suitable space. For each agent, we can consider the variance of the best estimator for yesterday's state based on observed actions. We can analyze these variances using the envelope theorem. Moreover, the space of these variances is a sufficient statistic for determining all agent strategies and action variances.

Let $r_{i,t}$ be i's *social signal*—the best estimator of θ_{t-1} based on the period $t-1$ actions of agents in N_i—and let $\kappa_{i,t}^2$ be the variance of $r_{i,t} - \theta_{t-1}$.

We claim that Φ induces a map $\widetilde{\Phi}$ on the space of variances $\kappa_{i,t}^2$, which we denote $\widetilde{\mathcal{V}}$. We must check the period t variances $(\kappa_{i,t}^2)_i$ uniquely determine all period $t+1$ variances $(\kappa_{i,t+1}^2)_i$: The variance $\boldsymbol{V}_{ii,t}$ of agent i's action, as well as the covariances $\boldsymbol{V}_{ii',t}$ of all pairs of agents i, i' with $N_i = N_{i'}$, are determined by $\kappa_{i,t}^2$. Moreover, by the condition on our network, these variances and covariances determine all agents' strategies in period $t+1$, and this is enough to pin down all period $t+1$ variances $\kappa_{i,t+1}^2$.

The proof proceeds by showing $\widetilde{\Phi}$ is a contraction on $\widetilde{\mathcal{V}}$ in the sup norm.

For each agent j, we have $N_i = N_{i'}$ for all $i, i' \in N_j$. So the period t actions of an agent i' in N_j are

$$a_{i',t} = \frac{(\rho^2\kappa_{i,t}^2 + 1)^{-1}}{\sigma_{i'}^{-2} + (\rho^2\kappa_{i,t}^2 + 1)^{-1}} \cdot r_{i,t} + \frac{\sigma_{i'}^{-2}}{\sigma_{i'}^{-2} + (\rho^2\kappa_{i,t}^2 + 1)^{-1}} \cdot s_{i',t} \tag{OA-3}$$

where $s_{i',t}$ is agent (i')'s signal in period t and $r_{i,t}$ the social signal of i (the same one that i' has). It follows from this formula that each action observed by j is a linear combination of a private signal and a *common* estimator $r_{i,t}$, with positive coefficients which sum to one. For simplicity we write

$$a_{i',t} = b_0 \cdot r_{i,t} + b_{i'} \cdot s_{i',t} \tag{OA-4}$$

(where b_0 and $b_{i'}$ depend on i' and t, but we omit these subscripts). We will use the facts $0 < b_0 < 1$ and $0 < b_{i'} < 1$.

We are interested in how $\kappa_{j,t+1}^2 = \text{Var}(r_{j,t+1} - \theta_t)$ depends on $\kappa_{i,t}^2 = \text{Var}(r_{i,t} - \theta_{t-1})$. The estimator $r_{j,t+1}$ is a linear combination of observed actions $a_{i',t}$, and therefore can be expanded as a linear combination of signals $s_{i',t}$ and the estimator $r_{i,t}$. We can write

$$r_{j,t+1} = c_0 \cdot (\rho r_{i,t}) + \sum_{i'} c_{i'} s_{i',t} \tag{OA-5}$$

and therefore (taking variances of both sides)

$$\begin{aligned} \kappa_{j,t+1}^2 = \text{Var}(r_{j,t+1} - \theta_t) &= c_0^2 \text{Var}(\rho r_{i,t} - \theta_t) + \sum_{i'} c_{i'}^2 \sigma_{i'}^2 \\ &= c_0^2(\rho^2\kappa_{i,t}^2 + 1) + \sum_{i'} c_{i'}^2 \sigma_{i'}^2 \end{aligned}$$

The desired result, that $\widetilde{\Phi}$ is a contraction, will follow if we can show that the derivative $\frac{d\kappa_{j,t+1}^2}{d\kappa_{i,t}^2} = c_0^2\rho^2 \in [0, \delta]$ for some $\delta < 1$. By the envelope theorem, when calculating this derivative, we can assume that the weights placed on actions $a_{i',t}$ by the estimator $r_{j,t+1}$ do not change as we vary $\kappa_{i,t}^2$, and therefore c_0 and the $c_{i'}$ above do not change. So it is enough to show the coefficient c_0 is in $[0, 1]$.

The intuition for the lower bound is that *anti-imitation* (agents placing negative weights on observed actions) only occurs if observed actions put too much weight on public information. But if $c_0 < 0$, then the weight on public information is actually negative so there is no reason to anti-imitate. This is formalized in the following lemma.

Lemma OA2. Suppose j has symmetric neighbors. Then the social signal of an agent at node j places nonnegative weight on a neighbor i's social signal from the previous period, i.e., $c_0 \geq 0$.

Proof. To check this formally, suppose that c_0 is negative. Then the social signal $r_{j,t+1}$ puts negative weight on some observed action—say the action $a_{k,t}$ of agent k. We want to check that the covariance of $r_{j,t+1} - \theta_t$ and $a_{k,t} - \theta_t$ is negative. Using (OA-4) and (OA-5), we compute that

$$\begin{aligned}\mathrm{Cov}(r_{j,t+1} - \theta_t, a_{k,t} - \theta_t) &= \mathrm{Cov}\left(c_0(\rho r_{i,t} - \theta_t) + \sum_{i' \in N_j} c_{i'}(s_{i',t} - \theta_t)), b_0(\rho r_{i,t} - \theta_t) + b_k(s_{k,t} - \theta_t)\right) \\ &= c_0 b_0 \mathrm{Var}(\rho r_{i,t} - \theta_t) + c_k b_k \mathrm{Var}(s_{k,t} - \theta_t)\end{aligned}$$

because all distinct summands above are mutually independent. We have $b_0, b_k > 0$, while $c_0 < 0$ by assumption and $c_k < 0$ because the estimator $r_{j,t+1}$ puts negative weight on $a_{k,t}$. So the expression above is negative. Therefore, it follows from the usual Gaussian Bayesian updating formula that the best estimator of θ_t given $r_{j,t+1}$ and $a_{k,t}$ puts positive weight on $a_{k,t}$. However, this is a contradiction: the best estimator of θ_t given $r_{j,t+1}$ and $a_{k,t}$ is simply $r_{j,t+1}$, because $r_{j,t+1}$ was defined as the best estimator of θ_t given observations that included $a_{k,t}$. This completes the proof of Lemma OA2. □

We now complete the proof of Lemma OA1. For the upper bound $c_0 \leq 1$, the idea is that $r_{j,t+1}$ puts more weight on agents with better signals while these agents put little weight on public information, which keeps the overall weight on public information from growing too large.

Note that $r_{j,t+1}$ is a linear combination of actions $\rho a_{i',t}$ for $i' \in N_j$, with coefficients summing to 1. The only way the coefficient on $\rho r_{i,t}$ in $r_{j,t+1}$ could be at least 1 would be if some of these coefficients on $\rho a_{i',t}$ were negative and the estimator $r_{j,t+1}$ placed greater weight on actions $a_{i',t}$ which placed more weight on $r_{i,t}$.

Applying the formula (OA-3) for $a_{i',t}$, we see that the coefficient b_0 on $\rho r_{i,t}$ is less than 1 and increasing in $\sigma_{i'}$. On the other hand, it is clear that the weight on $a_{i',t}$ in the social signal $r_{j,t+1}$ is decreasing in $\sigma_{i'}$: more weight should be put on more precise individuals. So in fact the estimator $r_{j,t+1}$ places less weight on actions $a_{i',t}$ which placed more weight on $r_{i,t}$.

Moreover, the coefficients placed on private signals are bounded below by a positive constant when we restrict to covariances in the image of $\widetilde{\Phi}$ (because all covariances are bounded as in the proof of Proposition 1). Therefore, each agent (i', t) with $i' \in N_j$ places weight at most one on the estimator $\rho r_{i,t-1}$. Agent j's social signal $r_{j,t+1}$ is a sum of these agents' actions with coefficients summing to 1 and satisfying the monotonicity property above. We conclude that the coefficient on $\rho r_{i,t}$ in the expression for $r_{j,t+1}$ is at most one. This completes the proof of Lemma OA1. □

We now prove Proposition 2.

Proof of Proposition 2. By Lemma OA1 there is a unique equilibrium on any network G with symmetric neighbors. Let $\varepsilon > 0$.

Consider any agent (i, t). Her neighbors have the same private signal qualities and the same neighborhoods (by the symmetric neighbors assumption). So there exists an equilibrium where for all i, the actions of agent (i,t)'s neighbors are exchangeable. By uniqueness, this in fact holds at the sole equilibrium.

So agent (i,t)'s social signal is an average of her neighbors' actions:

$$r_{i,t} = \frac{1}{|N_i|}\sum_{j\in N_i} a_{j,t-1}.$$

Suppose the ε-aggregation benchmark is achieved. Then all agents must place weight at least $\frac{(1+\varepsilon)^{-1}}{(1+\varepsilon)^{-1}+\sigma^{-2}}$ on their social signals. So at time t, the social signal $r_{i,t}$ places weight at least $\frac{(1+\varepsilon)^{-1}}{(1+\varepsilon)^{-1}+\sigma^{-2}}$ on signals from $t-2$ or earlier. Let Y be any linear combination of signals from $t-2$ or earlier with weights summing to 1. Then $\mathbb{E}[(Y-\theta_{t-1})^2] \geq 1$.[10] It follows that for ε sufficiently small the social signal $r_{i,t}$ is bounded away from a perfect estimate of θ_{t-1}. This gives a contradiction. □

OA4.3. **Proof of Corollary 1.** Consider a complete graph in which all agents have signal variance σ^2 and memory $m = 1$. By Proposition 2, as n grows large the variances of all agents converge to $A > (1+\sigma^{-2})^{-1}$.

Choose σ^2 large enough such that $A > 1$. To see that we can do this, note that as σ^2 grows large, the weight each agent places on their private signal vanishes. So the weight on signals from at least k periods ago approaches one for any k. Taking σ^2 such that this holds for k sufficiently large, we have $A > 1$.

Now suppose that we increase σ_1^2 to ∞. Then $a_{1,t} = r_{1,t}$ in each period, so all agents can infer all private signals from the previous period. As n grows large, the variance of agent 1 converges to 1 and the variances of all other agents converge to $(1+\sigma^{-2})^{-1}$. By our choice of σ^2, this gives a Pareto improvement. We can see by continuity that the same argument holds for σ_1^2 finite but sufficiently large.

OA4.4. **Proof of Corollary 2.** Our goal is to estimate $\mathrm{Var}(a_{i,t}-\theta_t)$.

First, observe that

$$\begin{aligned} a_{i,t} &= w_s s_{i,t} + (1-w_s)\frac{1}{n}\sum_j a_{j,t-1} \\ &= w_s(\theta_t + \varepsilon_{i,t}) + (1-w_s)\frac{1}{n}\sum_j a_{j,t-1}. \end{aligned}$$

This implies, inductively, that

$$a_{i,t} - \theta_t = w_s\varepsilon_{i,t} + \sum_{\ell=1}^{\infty}\left(w_s(1-w_s)^{\ell}(\theta_t - \theta_{t-\ell} + \zeta_t)\right),$$

where the ζ_t are mean-zero random variables independent of all other random variables in the expression. (They are linear combinations of agents' signal noise realizations.) Thus,

$$\mathrm{Var}(a_{i,t}-\theta_t) \geq \mathrm{Var}\left(w_s\sum_{\ell=1}^{\infty}(1-w_s)^{\ell}(\theta_t-\theta_{t-\ell})\right)$$

[10] This is because $\rho\theta_{t-2}$ is a sufficient statistic and an unbiased estimator for θ_{t-1} given signals up to $t-2$, and $\mathbb{E}[(\rho\theta_{t-2}-\theta_{t-1})^2] = 1$.

Noting that $\theta_t = \sum_{k=0}^{\infty} \rho^k \nu_{t-k}$, we may write

$$\theta_t - \theta_{t-\ell} = \sum_{k=0}^{\ell-1} \rho^k \nu_{t-k}$$

and therefore

$$\begin{aligned}
\mathrm{Var}(a_{i,t} - \theta_t) &\geq \mathrm{Var}\left(w_s \sum_{\ell=1}^{\infty} (1-w_s)^\ell \sum_{k=0}^{\ell-1} \rho^k \nu_{t-k} \right) \\
&= w_s^2 \mathrm{Var}\left(\sum_{k=0}^{\infty} \nu_{t-k} \rho^k \sum_{\ell=k+1}^{\infty} (1-w_s)^\ell \right) \\
&= w_s^2 \mathrm{Var}\left(\sum_{k=0}^{\infty} \nu_{t-k} \left(\rho^k \sum_{\ell=k+1}^{\infty} (1-w_s)^\ell \right) \right) \\
&= w_s^2 \sum_{k=0}^{\infty} \left(\rho^k \sum_{\ell=k+1}^{\infty} (1-w_s)^\ell \right)^2 \\
&= \frac{(1-w_s)^2}{1-(1-w_s)^2\rho^2}.
\end{aligned}$$

This proves the bound

$$\mathrm{Var}(a_{i,t} - \theta_t) \geq \frac{(1-w_s)^2}{1-(1-w_s)^2\rho^2}.$$

It remains to show the variances diverge to infinity as $\sigma^2 \to \infty$ and $\rho \to 1$ from below. Choose a sequence of pairs $(\sigma^2, \rho) \to (\infty, 1)$. If $w_s \to 0$ along any subsequence of this sequence, then along the subsequence we have $\frac{(1-w_s)^2}{1-(1-w_s)^2\rho^2} \to \infty$ and so $\mathrm{Var}(a_{i,t} - \theta_t) \to \infty$ as well. If w_s is nonvanishing, then $\mathrm{Var}(a_{i,t} - \theta_t) \to \infty$ since the action variance is at least $w_s^2\sigma^2$ and $\sigma^2 \to \infty$. Finally, note that these bounds are both independent of n, so $\mathrm{Var}(a_{i,t} - \theta_t) \to \infty$ uniformly in n.

OA4.5. **Proof of Theorem 2.** Suppose that all private signals have variance $\sigma^2 > 0$. Fix a sequence of networks G_n and an equilibrium on each G_n. We will show that given any constant C > 0 and any sequence of equilibria, the fraction of agents i such that

$$\widehat{\kappa}_i^2 \leq \frac{C}{d}$$

is bounded away from one.

We first prove the result in the case $m = 1$. For each n, let $\mathcal{G}_n$ be the set of agents i satisfying

$$\widehat{\kappa}_i^2 \leq \frac{C}{d},$$

i.e., the set of agents who do learn well. Assume for the sake of contradiction that $\frac{|\mathcal{G}_n|}{n} \to 1$ as $n \to \infty$ along some subsequence and pass to that subsequence.

For each j, we can express the action $a_{j,t}$ as a weighted sum of innovations and signal errors,[11] with all terms on the right-hand side conditionally independent:

$$a_{j,t} = \theta_t - \sum_{l=0}^{\infty} w_{j,t}(\nu_{t-l})(\rho^l \nu_{t-l}) + \sum_{l,j'} w_{j,t}(\eta_{j',t-l})(\rho^l \eta_{j',t-l}).$$

Here we use arguments on w to indicate the exogenous random variable that the coefficient pertains to. The coefficients in the expression above are uniquely determined.

Lemma OA3. For all $j \in \mathcal{G}_n$ we must have

$$w_{j,t}(\nu_t) \in \left(\frac{1}{\sigma^{-2}+1} - \frac{C'}{\overline{d}}, \frac{1}{\sigma^{-2}+1} \right)$$

for some $C' > 0$ (independent of j and n).

Proof. By the standard updating formula, the optimal weight $w_{j,t}(\nu_t)$ is $\frac{(\rho^2\kappa_{j,t}^2+1)^{-1}}{(\rho^2\kappa_{j,t}^2+1)^{-1}+\sigma^{-2}}$, where $\kappa_{j,t}^2$ is the variance of the best estimator of θ_{t-1} based on (j,t)'s social observations. The upper bound follows because this is minimized when $\kappa_{j,t}^2 = 0$. For the lower bound,

$$\begin{aligned} w_{j,t}(\nu_t) &= \frac{(\rho^2\kappa_{j,t}^2+1)^{-1}}{(\rho^2\kappa_{j,t}^2+1)^{-1}+\sigma^{-2}} \\ &= \frac{1}{(1+\sigma^{-2})+\sigma^{-2}\rho^2\kappa_{j,t}^2} \\ &= \frac{1}{1+\sigma^{-2}} - \frac{\sigma^{-2}\rho^2}{(1+\sigma^{-2})^2}\kappa_{j,t}^2 + O(\kappa_{j,t}^4). \end{aligned}$$

For $\kappa_{j,t}^2$ in any neighborhood of zero, we can choose C'' such that the nonconstant terms in the final expression are bounded below by $-C''\kappa_{j,t}^2$. Since by assumption we have $\kappa_{j,t}^2 \le \frac{C}{\overline{d}}$, the lemma follows with $C' = C \cdot C''$. □

There are at most $(n - |\mathcal{G}_n|)\overline{d}$ links directed to to agents outside $\mathcal{G}_n$. Each agent who observes at least $\frac{2(n-|\mathcal{G}_n|)}{n} \cdot \overline{d}$ agents outside $\mathcal{G}_n$ accounts for at least $\frac{2(n-|\mathcal{G}_n|)}{n} \cdot \overline{d}$ of those links, so there can be at most $\frac{n}{2}$ such agents. Since $\frac{|\mathcal{G}_n|}{n} \to 1$, there is an agent $i \in \mathcal{G}_n$ who observes fewer than $\frac{2(n-|\mathcal{G}_n|)}{n} \cdot \overline{d}$ agents outside $\mathcal{G}_n$.

Consider the action of such an agent $i \in \mathcal{G}_n$ in period $t+1$. Since

$$\kappa_{i,t+1}^2 \le \frac{C}{\overline{d}},$$

the weight on the innovation ν_t from the *previous* period t satisfies

$$\big(w_{i,t+1}(\nu_t)\rho\big)^2 \le \frac{C}{\overline{d}}. \tag{OA-6}$$

[11] To simplify calculations, we write this expression with a negative coefficient on the first sum so that the terms $w_{j,t}(\nu_{t-l})$ are positive. The weight that j places on ν_{t-l} is in fact $-w_{j,t}(\nu_{t-l})$.

On the other hand, we can express this weight in terms of neighbors' weights as

$$w_{i,t+1}(\nu_t) = \sum_j \rho w_{ij,t+1} w_{j,t}(\nu_t).$$

We will show that if this weight $w_{i,t+1}(\nu_t)$ vanishes, then the contribution of private signal errors to $\kappa_{i,t+1}$ must grow asymptotically faster than $1/\overline{d}$.

We can split this summation as

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + \rho \sum_{j \notin \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t).$$

We now consider two cases, depending on whether $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$, i.e., whether the sum of the absoulte values of the weights on agents outside $\mathscr{G}_n$ is vanishing.

Case 1: $\liminf_n \sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| = 0$. We can pass to a subsequence along which $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$.

We claim that it follows from the bounds on $w_{j,t}(\nu_t)$ in Lemma OA3 that this can only occur if $\sum_j |w_{ij,t+1}| \to \infty$. If $\sum_j |w_{ij,t+1}|$ is bounded,

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + \rho \sum_{j \notin \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + o(1).$$

The second equality holds because $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$ and $w_{j,t}(\nu_t) \in [0,1]$ for all j. Therefore,

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + o(1) \geq \left(\rho \frac{1}{1+\sigma^{-2}} \frac{\sigma^{-2}}{\sigma^{-2}+1} - o\,(1) \right) + o(1),$$

and the right-hand side is nonvanishing. Here the first term on the right-hand side is the limit of the sum if all of the terms $w_{j,t}(\nu_t)$ were equal to the upper bound $\frac{\sigma^{-2}}{\sigma^{-2}+1}$. The first $o(1)$ error term corresponds to the variation in $w_{j,t}(\nu_t)$ across j, which is $O(\frac{1}{\overline{d}})$ by Lemma OA3 and has bounded coefficients. Thus $w_{i,t+1}(\nu_t)$ is nonvanishing, but this contradicts the inequality (OA-6). We have proven the claim.

The contribution to $\kappa_{i,t+1}^2$ from signal errors $\eta_{j,t}$ is $\sum_j |w_{ij,t+1}|^2 (w_{j,t}^s)^2 \sigma^2$. Since $w_{j,t}^s = 1 - w_{j,t}(\nu_t)$ converge uniformly to a constant $\frac{\sigma^{-2}}{\sigma^{-2}+1}$, we can bound this contribution below by an expression that is proportional to

$$\sum_j |w_{ij,t+1}|^2.$$

The summation has at most $\overline{d}$ nonzero terms. Applying the standard bound $\|v\|_1 \leq \sqrt{n}\|v\|_2$ on L^p norms on $\mathbb{R}^n$,

$$\sum_j |w_{ij,t+1}|^2 \geq \frac{1}{\overline{d}} \left(\sum_j |w_{ij,t+1}| \right)^2.$$

The right-hand side of this inequality grows at a rate faster than $\frac{1}{\overline{d}}$ by the claim $\sum_j |w_{ij,t+1}| \to \infty$, and so the social signal error grows at a rate faster than $\frac{1}{\overline{d}}$. This gives a contradiction.

Case 2: $\liminf_n \sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| > 0$.

As in Case 1, the contribution to signal errors from neighbors $j \notin \mathscr{G}_n$ is proportional to

$$\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}|^2.$$

By our choice of the agent i, she observes at most $\frac{2(n-|\mathscr{G}_n|)}{n} \cdot \overline{d}$ agents outside $\mathscr{G}_n$. The same standard bound on L^p norms gives

$$\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}|^2 \geq \frac{1}{\overline{d}} \cdot \frac{n}{2(n-|\mathscr{G}|_n)} \left(\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \right)^2.$$

By assumption, the cardinality $n - |\mathscr{G}_n|$ of the complement of $\mathscr{G}_n$ is $o(n)$ and $(\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}|)^2$ is nonvanishing. So the right-hand side grows at a rate faster than $\frac{1}{\overline{d}}$. Thus the social signal error grows at a rate faster than $\frac{1}{\overline{d}}$, which again gives a contradiction. This completes the proof in the case $m = 1$, and we next turn to the general argument.

Now, suppose $m \geq 1$ is arbitrary. As before, for each agent (j, t),we can write:

$$a_{j,t} = \theta_t - \sum_{l=0}^{\infty} w_{j,t}(\nu_{t-l})(\rho^l \nu_{t-l}) + \sum_{l,j'} w_{j,t}(\eta_{j',t-l})(\rho^l \eta_{j',t-l}).$$

For each n, let $\mathscr{G}_n$ be the set of i satisfying

$$\widehat{\kappa}_i^2 \leq \frac{C}{\overline{d}}.$$

Suppose $\limsup_n |\mathscr{G}_n|/n = 1$. Passing to a subsequence, we can assume that $\lim_n |\mathscr{G}_n|/n = 1$, i.e., the fraction of agents in $\mathscr{G}_n$ converges to one.

As in the $m = 1$ proof above, we can choose $i \in \mathscr{G}_n$ who observes fewer than $\frac{2(n-|\mathscr{G}_n|)}{n} \cdot \overline{d}$ agents outside $\mathscr{G}_n$. Choose any such i and consider the agent (i, t) with $i \in \mathscr{G}_n$, who observes neighbors' actions in periods $t - 1, \ldots, t - m$. For each $1 \leq l \leq m$, we will write $w_{(i,t),(j,t-l)}$ for the weight that agent (i, t) places on the action of agent $(j, t - l)$. By the same argument as in Case 2 of the $m = 1$ proof above, $\liminf_n \sum_{j \notin \mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$ for each l (since the fraction of agents outside $\mathscr{G}_n$ is vanishing). Passing to a subsequence, we can assume that $\lim_n \sum_{j \notin \mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$.

We can express agent (i, t)'s action:

$$a_{i,t} = \sum_{1 \leq l \leq m} \left(\sum_{j \in \mathscr{G}_n} w_{(i,t),(j,t-l)} \rho^l a_{j,t-l} + \sum_{j \notin \mathscr{G}_n} w_{(i,t),(j,t-l)} \rho^l a_{j,t-l} \right).$$

We will show that this expression places nonvanishing weight on the innovation ν_{t-l} for some $l \geq 1$. This will contradict our assumption that $i \in \mathscr{G}_n$.

Since $\lim_n \sum_{j \notin \mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$ and the weight each agent places on ν_{t-l} is bounded, it is sufficient to show that

$$\sum_{1 \leq l \leq m} \sum_{j \in \mathscr{G}_n} w_{(i,t),(j,t-l)} \rho^l a_{j,t-l}$$

places nonvanishing weight on the innovation ν_{t-l} for some $l \geq 1$.

For each (j, t') such that $j \in \mathscr{G}_n$, we have

$$a_{j,t'} = \frac{\theta_{t'-1} + \sigma^{-2} s_{j,t'}}{1+\sigma^{-2}} + \epsilon_{j,t'},$$

where $\mathrm{Var}(\epsilon_{j,t'}) \to 0$. This is because

$$a_{j,t'} = \frac{(\rho^2\kappa_{i,t}^2 + 1)^{-1} r_{i,t} + \sigma^{-2} s_{j,t'}}{(\rho^2\kappa_{i,t}^2 + 1)^{-1} + \sigma^{-2}},$$

and we have $\kappa_{i,t}^2 = \mathrm{Var}(r_{i,t} - \theta_{t-1}) \to 0$.

Using this expression for $a_{j,t'}$, we obtain

$$\sum_{1\le l\le m} \sum_{j \in \mathcal{G}_n} w_{(i,t),(j,t-l)} \rho^l a_{j,t-l} = \sum_{1\le l\le m} \sum_{j \in \mathcal{G}_n} w_{(i,t),(j,t-l)} \rho^l \left(\frac{\theta_{t-l-1} + \sigma^{-2} s_{j,t-l}}{1+\sigma^{-2}} + \epsilon_{j,t-l} \right)$$

By the same argument as in Case 1 of the $m = 1$ proof above,

$$\sum_{1\le l\le m} \sum_{j \in \mathcal{G}_n} |w_{(i,t),(j,t-l)}|$$

must be bounded (or else the contributions of signal errors to $\widehat{\kappa}_i^2$ would be too large to have $i \in \mathcal{G}_n$). Therefore, it is sufficient to show that

$$\sum_{1\le l\le m} \sum_{j \in \mathcal{G}_n} w_{(i,t),(j,t-l)} \rho^l \cdot \frac{\theta_{t-l-1} + \sigma^{-2} s_{j,t-l}}{1+\sigma^{-2}}$$

places nonvanishing weight on the innovation ν_{t-l} for some $l \geq 1$.

This holds for the largest l such that $\sum_{j \in \mathcal{G}_n} w_{(i,t),(j,t-l)}$ is nonvanishing. Such an l must exist, because

$$\sum_{1\le l\le m} \sum_{j \in \mathcal{G}_n} w_{(i,t),(j,t-l)} \to \frac{1}{1+\sigma^{-2}}$$

since $i \in \mathcal{G}_n$.

OA4.6. **Proof of Proposition 3.** For each agent i, we can write

$$a_{i,t} = w_i^s s_{i,t} + \sum_j W_{ij} \rho a_{j,t-1} = w_i^s s_{i,t} + \sum_j W_{ij} \left(\rho w_j^s s_{j,t} + \sum_{j'} W_{jj'} \rho a_{j',t-2} \right).$$

Because we assume $w_i^s < \overline{w} < 1$ and $w_j^s < \overline{w} < 1$ for all j, the total weight $\sum_{j,j'} W_{ij} W_{jj'} \rho$ on terms $a_{j',t-2}$ is bounded away from zero. Because the error variance of each of these terms is greater than 1, this implies agent i fails to achieve the ε-aggregation benchmark for $\varepsilon > 0$ sufficiently small.

OA4.7. **Proof of Proposition 4.** We prove the following statement, which includes the proposition as special cases.

Proposition OA1. Suppose the network G is strongly connected.[12] Consider weights $\boldsymbol{W}$ and $\boldsymbol{w}^s$ and suppose they are all positive, with an associated steady state $\boldsymbol{V}_t$. Suppose either

(1) there is an agent i whose weights are a Bayesian best response to $\boldsymbol{V}_t$, and some agent observes that agent and at least one other neighbor; or

[12] That is, there is a directed path from each node to each other node.

(2) there is an agent whose weights are a naive best response to $\boldsymbol{V}_t$, and who observes multiple neighbors.
Then the steady state $\boldsymbol{V}_t$ is Pareto-dominated by another steady state.

We provide the proof in the case $m = 1$ to simplify notation. The argument carries through with arbitrary finite memory.

Case (1): Consider an agent l who places positive weight on a rational agent k and positive weight on at least one other agent. Define weights $\overline{W}$ by $\overline{W}_{ij} = W_{ij}$ and $\overline{w}_i^s = w_i^s$ for all $i \neq k$, $\overline{W}_{kj} = (1-\epsilon)W_{kj}$ for all $j \leq n$, and $\overline{w}_k^s = (1-\epsilon)w_k^s + \epsilon$, where W_{ij} and w_i^s are the weights at the initial steady state. In words, agent k places weight $(1-\epsilon)$ on her equilibrium strategy and extra weight ϵ on her private signal. All other players use the same weights as at the steady state.

Suppose we are at the initial steady state until time t, but in period t and all subsequent periods agents instead use weights $\overline{W}$. These weights give an alternate updating function $\overline{\Phi}$ on the space of covariance matrices. Because the weights $\overline{W}$ are positive and fixed, all coordinates of $\overline{\Phi}$ are increasing, linear functions of all previous period variances and covariances. Explicitly, the diagonal terms are

$$[\overline{\Phi}(\boldsymbol{V}_t)]_{ii} = (\overline{w}_i^s)^2\sigma_i^2 + \sum_{j,j' \leq n} \overline{W}_{ij}\overline{W}_{ij'}(\rho^2 V_{jj',t} + 1)$$

and the off-diagonal terms are

$$[\overline{\Phi}(\boldsymbol{V}_t)]_{ii'} = \sum_{j,j' \leq n} \overline{W}_{ij}\overline{W}_{i'j'}(\rho^2 V_{jj,t'} + 1).$$

So it is sufficient to show the variances $\overline{\Phi}^h(\boldsymbol{V}_t)$ after applying $\overline{\Phi}$ for h periods Pareto dominate the variances in $\boldsymbol{V}_t$ for some h.

In period t, the change in weights decreases the covariance $V_{jk,t}$ of k and some other agent j, who l also observes, by $f(\epsilon)$ of order $\Theta(\epsilon)$. By the envelope theorem, the change in weights only increases the variance V_{kk} by $O(\epsilon^2)$. Taking ϵ sufficiently small, we can ignore $O(\epsilon^2)$ terms.

There exists a constant $\delta > 0$ such that all initial weights on observed neighbors are at least δ. Then each coordinate $[\Phi(\boldsymbol{V})]_{ii}$ is linear with coefficient at least δ^2 on each variance or covariance of agents observed by i.

Because agent l observes k and another agent, agent l's variance will decrease below its equilibrium level by at least $\delta^2 f(\epsilon)$ in period $t+1$. Because $\overline{\Phi}$ is increasing in all entries and we are only decreasing covariances, agent l's variance will also decrease below its initial level by at least $\delta^2 f(\epsilon)$ in all periods $t' > t+1$.

Because the network is strongly connected and finite, the network has a diameter. After $d+1$ periods, the variances of all agents have decreased by at least $\delta^{2d+2} f(\epsilon)$ from their initial levels. This gives a Pareto improvement.

Case (2): Consider a naive agent k who observes at least two neighbors. We can write agent k's period t action as

$$a_{k,t} = w_k^s s_{i,t} + \sum_{j \in N_i} W_{kj}\rho a_{j,t-1}.$$

Define new weights $\overline{W}$ as in the proof of case (1). Because agent k is naive and the summation $\sum_{j \in N_i} W_{kj}\rho a_{j,t-1}$ has at least two terms, she believes the variance of this summation is smaller than its true value. So marginally increasing the weight on $s_{k,t}$ and decreasing the weight on this summation decreases her action variance. This deviation also decreases her covariance with any other agent. The remainder of the proof proceeds as in case (1).

OA4.8. **Proof of Proposition 5.** Suppose the social influence

$$\mathrm{SI}(i) = \sum_{j \in N} \sum_{k=1}^{\infty} \left(\rho^k \widehat{\boldsymbol{W}}^k \right)_{ji} \widehat{w}_i^s = \left[\mathbf{1}' \left(I - \rho \widehat{\boldsymbol{W}} \right)^{-1} - \mathbf{1}' \right]_i \widehat{w}_i^s$$

does not converge for some i. Then in particular, there exists j such that $\sum_{k=0}^{\infty} \left(\rho^\ell \widehat{W} \right)_{ji}^k \widehat{w}_i^s$ does not converge. We can write

$$a_{j,t} = \sum_{\ell=0}^{\infty} \sum_{j' \in N} \left(\rho^\ell \widehat{\boldsymbol{W}}^\ell \right)_{jj'} \widehat{w}_{j'}^s s_{j',t-\ell}.$$

This expression is the sum of

$$\sum_{\ell=0}^{\infty} \left(\rho \widehat{\boldsymbol{W}}^\ell \right)_{ji} \widehat{w}_i^s \eta_{i,t-\ell}$$

and independent terms corresponding to signal errors of agents other than i and changes in the state. Because $\sum_{\ell=0}^{\infty} \left(\rho^\ell \widehat{\boldsymbol{W}}^\ell \right)_{ji} \widehat{w}_i^s$ does not converge, the payoff to action $a_{j,t}$ must therefore be $-\infty$. But we showed in the proof of Proposition 1 that agent j's equilibrium payoff is at least $-\sigma_j^2$, which gives a contradiction.

Given convergence, the expression for $\mathrm{SI}(i)$ follows from the Neumann series identity

$$\sum_{k=0}^{\infty} \boldsymbol{M}^k = (\boldsymbol{I} - \boldsymbol{M})^{-1}.$$

OA4.9. **Proof of Proposition 6.** The social signal $r_{i,t}$ is the same for all agents, and we will refer to it as r_t. We can express the social signal as

$$r_t = w_A \sum_{i:\sigma_i = \sigma_A} a_{i,t-1} + w_B \sum_{i:\sigma_i = \sigma_B} a_{i,t-1} \tag{OA-7}$$

for some weights w_A and w_B.

We can rewrite the actions $a_{i,t-1}$ for i with signal variance σ_A^2 as

$$a_{i,t-1} = \frac{K}{K + \sigma_A^{-2}} \rho r_{i,t-1} + \frac{\sigma_A{-2}}{K + \sigma_A^{-2}} s_{i,t-1},$$

where $K = \rho^2 \kappa_{t-1}^2 + 1$ is the equilibrium variance of ρr_{t-1} about the state θ_{t-1}. The analogous formula holds for agents i with signal variance σ_B^2.

Substituting the formulas for $a_{i,t-1}$ into equation (OA-7) and taking variances,

$$\begin{aligned}\kappa_t^2 &= \mathrm{Var}(r_t - \theta_{t-1}) \\ &= \mathrm{Var}\left(\frac{nK}{2}\left(\frac{w_A}{K+\sigma_A^{-2}} + \frac{w_B}{K+\sigma_B^{-2}}\right)(r_{t-1}-\theta_{t-1})\right) + \mathrm{Var}\left(w_A \sum_{i\in S_A} \frac{\sigma_A^{-2}}{K+\sigma_A^{-2}}(s_{i,t-1}-\theta_{t-1})\right) \\ &\qquad + \mathrm{Var}\left(w_B \sum_{i\in S_B} \frac{\sigma_B^{-2}}{K+\sigma_B^{-2}}(s_{i,t-1}-\theta_{t-1})\right) \\ &= \frac{n^2K}{4}\left(\frac{w_A}{K+\sigma_A^{-2}} + \frac{w_B}{K+\sigma_B^{-2}}\right)^2 + \frac{n}{2}\cdot\frac{w_A^2\sigma_A^{-2}}{(K+\sigma_A^{-2})^2} + \frac{n}{2}\cdot\frac{w_B^2\sigma_B^{-2}}{(K+\sigma_B^{-2})^2}.\end{aligned}$$

The equilibrium weights $\widehat{w}_A$ and $\widehat{w}_B$ minimize this expression. Using the fact that $\widehat{w}_A + \widehat{w}_B = \frac{2}{n}$, we have that $\widehat{w}_A$ satisfies

$$\begin{aligned}(\kappa^2)'_{t+1}(\widehat{w}_A) = \frac{n^2K}{2}\left(\frac{\widehat{w}_A}{K+\sigma_A^{-2}} + \frac{\widehat{w}_B}{K+\sigma_B^{-2}}\right)\left(\frac{1}{K+\sigma_A^{-2}} - \frac{1}{K+\sigma_B^{-2}}\right) + \frac{n\widehat{w}_A\sigma_A^{-2}}{(K+\sigma_A^{-2})^2} - \frac{n\widehat{w}_B\sigma_B^{-2}}{(K+\sigma_B^{-2})^2} \\ = 0.\end{aligned}$$

This equation, along with $\widehat{w}_A + \widehat{w}_B = \frac{2}{n}$, allows us to explicitly solve for $\widehat{w}_A$ and $\widehat{w}_B$ in terms of k and exogenous variables. In particular, we get that

$$\frac{\widehat{w}_A}{\widehat{w}_B} = \left(\frac{K+\sigma_A^{-2}}{K+\sigma_B^{-2}}\right)\left(\frac{2+Kn\sigma_B^2 - K(n-2)\sigma_A^2}{2+Kn\sigma_A^2 - K(n-2)\sigma_B^2}\right). \tag{OA-8}$$

We now turn to analyzing social influences. Recall that

$$\mathrm{SI}(i) = \sum_{j=1}^{n}\sum_{k=1}^{\infty}\left(\rho\widehat{W}\right)_{ij}^{k}\widehat{w}_i^s. \tag{OA-9}$$

On the complete graph, this expression is proportional to the product of the weight placed on agent i by the social signal r_t and agent i's self-weight w_i^s. Therefore, we compute

$$\frac{\mathrm{SI}(A)}{\mathrm{SI}(B)} = \frac{\widehat{w}_A}{\widehat{w}_B}\cdot\frac{\frac{\sigma_A^{-2}}{K+\sigma_A^{-2}}}{\frac{\sigma_B^{-2}}{K+\sigma_B^{-2}}}.$$

Substituting from equation (OA-8),

$$\frac{\mathrm{SI}(A)}{\mathrm{SI}(B)} = \left(\frac{\sigma_A^{-2}}{\sigma_B^{-2}}\right)\left(\frac{2+Kn\sigma_B^2 - K(n-2)\sigma_A^2}{2+Kn\sigma_A^2 - K(n-2)\sigma_B^2}\right).$$

We want to show that the left-hand side is greater than $\frac{\sigma_A^{-2}}{\sigma_B^{-2}}$ whenever $\sigma_A^{-2} > \sigma_B^{-2}$, which is equivalent to showing

$$\frac{2+Kn\sigma_B^2 - K(n-2)\sigma_A^2}{2+Kn\sigma_A^2 - K(n-2)\sigma_B^2} > 1$$

whenever $\sigma_A^{-2} > \sigma_B^{-2}$ and this fraction is positive.

To see this, note that the difference between the numerator and denominator of the fraction is

$$\begin{aligned}\left(Kn\sigma_B^2 - K(n-2)\sigma_A^2\right) - \left(Kn\sigma_A^2 - K(n-2)\sigma_B^2\right) &= 2K(n-1)(\sigma_B^2 - \sigma_A^2) \\ &> 0\end{aligned}$$

as desired.

OA5. Model with a starting time

In introducing the model (Section 2), we made the set of time indices $\mathcal{T}$ equal to $\mathbb{Z}$, the set of all integers. Here we study the variant with an initial time period, $t = 0$: thus, we take $\mathcal{T}$ to be $\mathbb{Z}_{\geq 0}$, the nonnegative integers. This section shows that there is a unique equilibrium outcome. In large networks, a suitable analogue of Theorem 1 holds, with both aggregation quality and outcomes similar to those obtained there. Similarly, the negative result of Proposition 2 also has a counterpart in this model.

Let θ_0 be drawn according to the stationary distribution of the state process: $\theta_0 \sim \mathcal{N}\left(0, \frac{1}{1-\rho}\right)$. After this, the state random variables θ_t satisfy the AR(1) evolution

$$\theta_{t+1} = \rho\theta_t + \nu_{t+1},$$

where ρ is a constant with $0 < |\rho| < 1$ and $\nu_{t+1} \sim \mathcal{N}(0, \sigma_\nu^2)$ are independent innovations. Actions, payoffs, signals, and observations are the same as in the main model, with the obvious modification that in the initial periods, $t < m$, information sets are smaller as there are not yet prior actions to observe.[13] To save on notation, we write actions as if agents had an improper prior, understanding that the adjustment for actions taken under the natural prior $\theta_t \sim \mathcal{N}\left(0, \frac{1}{1-\rho}\right)$ is immediate.

In this model, there is a straightforward prediction of behavior. A Nash equilibrium here refers to an equilibrium of the game involving all agents (i, t) for all time indices in $\mathcal{T}$.

Fact OA1. In the model with $\mathcal{T} = \mathbb{Z}_{\geq 0}$, there is a unique Nash equilibrium, and it is in linear strategies. The initial generation ($t = 0$) plays a linear strategy based on private signals only. In any period $t > 0$, given linear strategies from prior periods, players' best responses are linear. For time periods $t > m$, we have

$$\boldsymbol{V}_t = \Phi(\boldsymbol{V}_{t-1}).$$

This fact follows from the observation that the initial ($t = 0$) generation faces a problem of forming a conditional expectation of a Gaussian state based on Gaussian signals, so their optimal strategies are linear. From then on, the analysis of Section 3.1 characterizes best-response behavior inductively. Note that for arbitrary environments, the fact does not imply that $\boldsymbol{V}_t$ must converge.

Our main purpose in this section is to give analogues of the main results on learning in large networks. We use the same definition of an environment—in terms of the distribution of networks and signals—as in Section 4.3. For simplicity, we work with $m = 1$, though the arguments for our positive result extend straightforwardly.

The analogue of Theorem 1 is:

[13] The actions for $t < 0$ can be set to arbitrary (commonly known) constants.

Theorem OA1. Consider the $\mathcal{T} = \mathbb{Z}_{\geq 0}$ model. If an environment satisfies signal diversity, there is $C > 0$ such that asymptotically almost surely $\widehat{\kappa}^2_{i,t} < C/n$ for all i at all times $t \geq 1$ in the unique Nash equilibrium.

In particular, this implies that the covariance matrix in each period $t \geq 1$ is very close (in the Euclidean norm) to the good-learning equilibrium from Theorem 1. We sketch the proof, which uses the material we developed in Appendix B. We define A_t as in that proof (Section B.1). Take a $\beta > 0$, to be specified later, and consider

$$\overline{\mathcal{W}} = \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}} \cup \widetilde{\Phi}\left(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}\right).$$

First, for large enough β, we have that $A_1 \in \overline{\mathcal{W}}$: In the unique Nash equilibrium, at $t = 1$, agents simply take weighted averages of their neighbors' signals, weighted by their precisions. So $A_1 \in \overline{\mathcal{W}}$ by the central limit theorem for β sufficiently large. Second, we use the previously established fact (recall Section B.2) that $\widetilde{\Phi}(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$ to deduce that $A_t \in \overline{\mathcal{W}}$ at all future times. Finally, we observe that $\overline{\mathcal{W}} \subseteq \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$ by construction.

Without signal diversity, the unique equilibrium can feature bad learning forever. The analogue of Proposition 2 is immediate. In graphs with symmetric neighbors, Φ is a contraction when $m = 1$. So iteration of it arrives at the unique fixed point, and thus a learning outcome far from the benchmark.

OA6. Naive Agents

In this section we provide rigorous detail for the analysis given in 5.1. We will describe outcomes with two signal types, σ_A^2 and σ_B^2.[14] We use the same random network model as in Section 4.4 and assume each network type contains equal shares of agents with each signal type.

We can define variances

$$V_A^{\infty} = \frac{\rho^2\kappa_t^2 + 1 + \sigma_A^{-2}}{\left(1 + \sigma_A^{-2}\right)^2}, \qquad V_B^{\infty} = \frac{\rho^2\kappa_t^2 + 1 + \sigma_B^{-2}}{\left(1 + \sigma_B^{-2}\right)^2} \tag{OA-1}$$

where

$$\kappa_t^{-2} = 1 - \frac{1}{(\sigma_A^{-2} + \sigma_B^{-2})}\left(\frac{\sigma_A^{-2}}{1 + \sigma_A^{-2}} + \frac{\sigma_B^{-2}}{1 + \sigma_B^{-2}}\right).$$

Naive agents' equilibrium variances converge to these values.

Proposition OA2. Under the assumptions in this subsection:

(1) There is a unique equilibrium on G_n.

(2) Given any $\delta > 0$, asymptotically almost surely all agents' equilibrium variances are within δ of V_A^{∞} and V_B^{∞}.

(3) There exists $\varepsilon > 0$ such that asymptotically almost surely the ε-aggregation benchmark is not achieved, and when $\sigma_A^2 = \sigma_B^2$ asymptotically almost surely all agents' variances are larger than V^{∞}.

Aggregating information well requires a sophisticated response to the correlations in observed actions. Because naive agents completely ignore these correlations, their learning outcomes are

[14]The general case, with many signal types, is similar.

poor. In particular their variances are larger than at the equilibria we discussed in the Bayesian case, even when that equilibrium is inefficient ($\sigma_A^2 = \sigma_B^2$).

When signal qualities are homogeneous ($\sigma_A^2 = \sigma_B^2$), we obtain the same limit on any network with enough observations. That is, on any sequence $(G_n)_{n=1}^{\infty}$ of (deterministic) networks with the minimum degree diverging to ∞ and any sequence of equilibria, the equilibrium action variances of all agents converge to V_A^{∞}.

OA6.1. **Proof of Proposition OA2.** We first check that there is a unique naive equilibrium. As in the Bayesian case, covariances are updated according to equations (3.3):

$$\boldsymbol{V}_{ii,t} = (w_{i,t}^s)^2\sigma_i^2 + \sum W_{ik,t}W_{ik',t}(\rho^2\boldsymbol{V}_{kk',t-1} + 1) \text{ and } \boldsymbol{V}_{ij,t} = \sum W_{ik,t}W_{i'k',t}(\rho^2\boldsymbol{V}_{kk',t-1} + 1).$$

The weights $W_{ik,t}$ and $w_{i,t}^s$ are now all positive constants that do not depend on $\boldsymbol{V}_{t-1}$. So differentiating this formula, we find that all partial derivatives are bounded above by $1 - w_{i,t}^s < 1$. So the updating map (which we call Φ^{naive}) is a contraction in the sup norm on $\mathcal{V}$. In particular, there is at most one equilibrium.

The remainder of the proof characterizes the variances of agents at this equilibrium. We first construct a candidate equilibrium with variances converging to V_A^{∞} and V_B^{∞}, and then we show that for n sufficiently large, there exists an equilibrium nearby in $\mathcal{V}$.

To construct the candidate equilibrium, suppose that each agent observes the same number of neighbors of each signal type. Then there exists an equilibrium $\widehat{\boldsymbol{V}}^{\text{sym}}$ where covariances depend only on signal types, i.e., $\widehat{\boldsymbol{V}}^{\text{sym}}$ is invariant under permutations of indices that do not change signal types. We now show variances of the two signal types at this equilibrium converge to V_A^{∞} and V_B^{∞}.

To estimate θ_{t-1}, a naive agent combines observed actions from the previous period with weight proportional to their precisions σ_A^{-2} or σ_B^{-2}. The naive agent incorrectly believes this gives an almost perfect estimate of θ_{t-1}. So the weight on older observations vanishes as $n \to \infty$. The naive agent then combines this estimate of θ_{t-1} with her private signal, with weights converging to the weights she uses if the estimate is perfect.

Agent i observes $\frac{|N_i|}{2}$ neighbors of each signal type, so her estimate $r_{i,t}^{\text{naive}}$ of θ_{t-1} is approximately:

$$r_{i,t}^{\text{naive}} = \frac{2}{|N_i|(\sigma_A^{-2} + \sigma_B^{-2})}\left[\sigma_A^{-2}\sum_{j\in N_i,\sigma_j^2=\sigma_A^2} a_{j,t-1} + \sigma_B^{-2}\sum_{j\in N_i,\sigma_j^2=\sigma_B^2} a_{j,t-1}\right].$$

The actual variance of this estimate converges to:

$$\operatorname{Var}(r_{i,t}^{\text{naive}} - \theta_{t-1}) = \frac{1}{(\sigma_A^{-2} + \sigma_B^{-2})}\left[\sigma_A^{-4}\operatorname{Cov}_{AA}^{\infty} + \sigma_B^{-4}\operatorname{Cov}_{BB}^{\infty} + 2\sigma_A^{-2}\sigma_B^{-2}\operatorname{Cov}_{AB}^{\infty}\right] \tag{OA-2}$$

where $\operatorname{Cov}_{AA}^{\infty}$ is the covariance of two distinct agents of signal type A and $\operatorname{Cov}_{BB}^{\infty}$ and $\operatorname{Cov}_{AB}^{\infty}$ are defined similarly.

Since agents believe this variance is close to 1, the action of any agent with signal variance σ_A^2 is approximately:

$$a_{i,t} = \frac{r_{i,t}^{\text{naive}} + \sigma_A^{-2} s_{i,t}}{1 + \sigma_A^{-2}}.$$

We can then compute the limits of the covariances of two distinct agents of various signal types to be:

$$\mathrm{Cov}_{AA}^{\infty} = \frac{\rho^2 \kappa_t^2 + 1}{\left(1 + \sigma_A^{-2}\right)^2}; \;\; \mathrm{Cov}_{BB}^{\infty} = \frac{\rho^2 \kappa_t^2 + 1}{\left(1 + \sigma_B^{-2}\right)^2}; \;\; \mathrm{Cov}_{AB}^{\infty} = \frac{\rho^2 \kappa_t^2 + 1}{\left(1 + \sigma_A^{-2}\right)\left(1 + \sigma_B^{-2}\right)}.$$

Plugging into OA-2 we obtain

$$\kappa_t^{-2} = 1 - \frac{1}{(\sigma_A^{-2} + \sigma_B^{-2})} \left(\frac{\sigma_A^{-2}}{1 + \sigma_A^{-2}} + \frac{\sigma_B^{-2}}{1 + \sigma_B^{-2}} \right).$$

Using this formula, we can check that the limits of agent variances in $\widehat{\boldsymbol{V}}^{\text{sym}}$ match equations OA-1.

We must check there is an equilibrium near $\widehat{\boldsymbol{V}}^{\text{sym}}$ with high probability. Let $\zeta = 1/n$. Let E be the event that for each agent i, the number of agents observed by i with private signal variance σ_A^2 is within a factor of $[1 - \zeta^2, 1 + \zeta^2]$ of its expected value, and similarly the number of agents observed by i with private signal variance σ_B^2 is within a factor of $[1 - \zeta^2, 1 + \zeta^2]$ of its expected value. This event implies that each agent observes a linear number of neighbors and observes approximately the same number of agents with each signal quality. We can show as in the proof of Theorem 1 that for n sufficiently large, the event E occurs with probability at least $1 - \zeta$. We condition on E for the remainder of the proof.

Let $\mathcal{V}_\varepsilon$ be the ε-ball around in $\widehat{\boldsymbol{V}}^{\text{sym}}$ the sup norm. We claim that for n sufficiently large, the updating map preserves this ball: $\Phi^{\text{naive}}(\mathcal{V}_\varepsilon) \subset \mathcal{V}_\varepsilon$. We have $\Phi^{\text{naive}}(\widehat{\boldsymbol{V}}^{\text{sym}}) = \widehat{\boldsymbol{V}}^{\text{sym}}$ up to terms of $O(1/n)$. As we showed in the first paragraph of this proof, the partial derivatives of Φ^{naive} are bounded above by a constant less than one. For n large enough, these facts imply $\Phi^{\text{naive}}(\mathcal{V}_\varepsilon) \subset \mathcal{V}_\varepsilon$. We conclude there is an equilibrium in $\mathcal{V}_\varepsilon$ by the Brouwer fixed point theorem.

Finally, we compare the equilibrium variances to the ε-aggregation benchmark and to V^∞. It is easy to see these variances are worse than the ε-aggregation benchmark for n large for some $\varepsilon > 0$, and therefore by Theorem 1 also asymptotically worse than the Bayesian case when $\sigma_A^2 \neq \sigma_B^2$.

In the case $\sigma_A^2 = \sigma_B^2$, it is sufficient to show that Bayesian agents place more weight on their private signals (since asymptotically action error comes from past changes in the state and not signal errors). Call the private signal variance σ^2. For Bayesian agents, we showed in Theorem 1 that the weight on the private signal is equal to $\frac{\sigma^{-2}}{\sigma^{-2} + (\rho^2 \mathrm{Cov}^\infty + 1)^{-1}}$ where Cov^∞ solves

$$\mathrm{Cov}^\infty = \frac{(\rho^2 \mathrm{Cov}^\infty + 1)^{-1}}{[\sigma^{-2} + (\rho^2 \mathrm{Cov}^\infty + 1)^{-1}]^2}.$$

For naive agents, the weight on the private signal is equal to $\frac{\sigma^{-2}}{\sigma^{-2}+1}$, which is smaller since $\mathrm{Cov}^\infty > 0$.

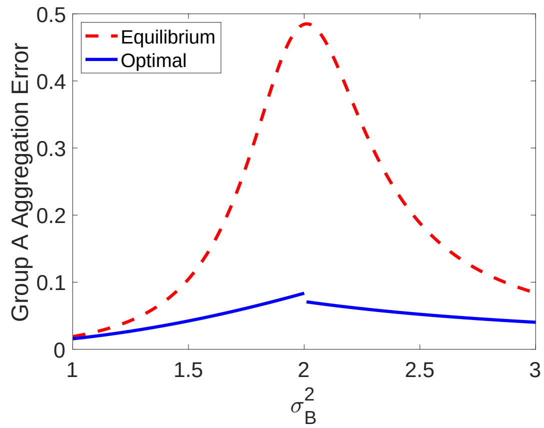

FIGURE OA7.1. Social planner's optimum and Bayesian learning. The red curve shows equilibrium aggregation errors on a complete graph with $n = 600$ agents, split into two equally-sized groups with private signal variances $\sigma_A^2 = 2$ and σ_B^2 varying. The blue curve plots the aggregation errors when weights are chosen by a social planner the sum of agents' steady-state action variances.

OA7. SOCIALLY OPTIMAL LEARNING OUTCOMES WITH NON-DIVERSE SIGNALS

In this section, we show that a social planner can achieve vanishing aggregation errors even when signals are non-diverse. Thus, slower rate of learning at equilibrium with non-diverse signals is a consequence of individual incentives rather than a necessary feature of the environment.

Let G_n be the complete network with n agents. Suppose that $\sigma_i^2 = \sigma^2$ for all i and $m = 1$.

Proposition OA3. Let $\varepsilon > 0$. Under the assumptions in this section, for n sufficiently large there exist weights weights $\boldsymbol{W}$ and $\boldsymbol{w}^s$ such that at the corresponding steady state on G_n, the ε-aggregation benchmark is achieved.

Proof. An agent with a social signal equal to θ_{t-1} would place weight $\frac{\sigma^{-2}}{\sigma^{-2}+1}$ on her private signal and weight $\frac{1}{\sigma^{-2}+1}$ on her social signal. Let $w_A^s = \frac{\sigma^{-2}}{\sigma^{-2}+1} + \delta$ and $w_B^s = \frac{\sigma^{-2}}{\sigma^{-2}+1} - \delta$, where we will take $\delta > 0$ to be small.

Assume that the first $\lfloor n/2 \rfloor$ agents place weight w_A^s on their private signals and weight $1 - w_A^s$ on a common social signal r_t we will define, while the remaining agents place weight w_B^s on their private signals and weight $1 - w_B^s$ on the social signal r_t. As in the proof of Theorem 2,

$$\frac{1}{\lfloor n/2 \rfloor} \sum_{j=1}^{\lfloor n/2 \rfloor} a_{j,t-1} = w_A^s \theta_{t-1} + (1 - w_A^s) r_{t-1} + O(n^{-1/2}),$$

$$\frac{1}{\lceil n/2 \rceil} \sum_{j=\lfloor n/2 \rfloor+1}^{n} a_{j,t-1} = w_B^s \theta_{t-1} + (1 - w_B^s) r_{t-1} + O(n^{-1/2}).$$

There is a linear combination of these summations equal to $\theta_{t-1} + O(n^{-1/2})$, and we can take r_t equal to this linear combination. Taking δ sufficiently small and then n sufficiently large, we find that ε-perfect aggregation is achieved. □

In Figure OA7.1, we consider equilibrium and socially optimal outcomes with $n = 600$. Half of agents are in group A, with signal variance $\sigma_A^2 = 2$, while the other half are in group B, with signal variance σ_B^2 changing. In blue we plot average equilibrium aggregation errors for group A. In green we plot the average aggregation errors of group A when a social planner minimizes the total action variance (of both groups). The weights that each agent puts on her own private signal and the other agents are set to depend only on the groups. Under these socially optimal weights agents learn very well, and heterogeneity in signal variances only has a small impact.

References

Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2013a): *The Diffusion of Microfinance*, Harvard Dataverse, `https://doi.org/10.7910/DVN/U3BIHX`.

——— (2013b): "The diffusion of microfinance," *Science*, 341, 1236498.

Dasaratha, K., B. Golub, and N. Hak (2023): "Learning from Neighbours about a Changing State," *Review of Economic Studies*, available at arXiv:1801.02042.

Kay, S. M. (1993): *Fundamentals of Statistical Signal Processing*, Prentice Hall PTR.

Manresa, E. (2013): "Estimating the structure of social interactions using panel data," Working Paper, CEMFI, Madrid.