

**SUPPLY NETWORK FORMATION AND FRAGILITY
SUPPLEMENTARY APPENDIX FOR ONLINE PUBLICATION**

MATTHEW ELLIOTT, BENJAMIN GOLUB, AND MATTHEW V. LEDUC

This document contains supporting material for the paper “Supply Network Formation and Fragility,” which herein we refer to as the “main paper” or simply “paper.”

SA1. FORMAL CONSTRUCTION OF THE SUPPLY NETWORK (FOR ONLINE PUBLICATION)

We now formally construct the random supply network which was introduced in Sections IA and IB in the paper; this construction provides the foundation for calculating the set of functional firms, etc.

Endow each \mathcal{V}_i with the Borel σ -algebra and a scaling of Lebesgue measure, denoted λ , and let \mathcal{V} be the disjoint union of these spaces, with total measure 1. Fix positive integers m and n , as well as a distribution μ over the nonnegative integers. A (symmetric) *potential supply network with parameters* m, n, μ is a random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ satisfying the following properties.

- Its nodes are the set \mathcal{V} .
- Edges are ordered pairs (v, v') where $v' = (j, f')$ for $j \in I(i)$ —the meaning is that v can potentially source from v' . We depict such an edge as an arrow from v to v' .
- The measure of nodes v with $d(v) = d'$ is $\mu(d')$.
- Consider any $v \in \mathcal{V}_i$ with $d(v) > 0$.
 - For each $j \in I(i)$ there are n edges (v, v') to n distinct varieties $v' \in \mathcal{V}_j$. For any variety, define its neighborhood $N_v = \{v' : (v, v') \in \mathcal{E}\}$.
 - The elements of $N_v \cap \mathcal{V}_j$ are independently drawn from an atomless distribution over \mathcal{V}_j conditioned on $d(v') = d(v) - 1$.
 - For any countable set of varieties $\widehat{\mathcal{V}}$, its neighborhoods $(N_v)_{v \in \widehat{\mathcal{V}}}$ are independent.

Now fix *relationship strengths* $(x_v)_{v \in \mathcal{V}}$.¹ Define \mathcal{G}' to be a random subgraph of \mathcal{G} in which each edge from a node v of positive depth is kept independently, with probability x_v . More formally, define for every edge vv' a random variable $O_{vv'} \in \{0, 1\}$ (whether the edge is operational) such that

- $\mathbf{P}[O_{vv'} = 1 \mid \mathcal{G}] = x_v$ for every $vv' \in \mathcal{G}$ and,
- for any countable subset E of edges in \mathcal{G} , the random variables $(O_e)_{e \in E}$ are independent conditional on \mathcal{G} .

The out-neighborhood of the depth-0 varieties in \mathcal{G}' is $\bigcup_{j \in I(i)} \mathcal{V}_j$, since these can source from anyone.

A subset $\widehat{\mathcal{V}} \subseteq \mathcal{V}$ is defined to be *consistent* if, for each $v \in \widehat{\mathcal{V}}$ the following holds: for each product j that $v = (i, f)$ requires as inputs ($\forall j \in I(i)$), there is an operational edge $(v, v') \in \mathcal{G}'$ with $v' \in \widehat{\mathcal{V}}$. There may be many consistent sets, but by Tarki’s theorem, there will be a maximal one, \mathcal{V}' , which is a superset of any other consistent set. For any given variety, this can be found by the simple iterative procedure in Section IC in the paper.

Since any countable set of edges is independent, we can make computations about the relevant marginal probabilities in our model (e.g., reliability of any variety) as we would if there only one tree.

SA2. PROOFS OF MAIN RESULTS

SA2.1. The reliability curve: Basic precipice results. The first building block of our analysis is characterizing the shape of the reliability curve $x \mapsto \rho(x, \mu_\tau)$ when τ is large (so that depths are large). In Figure 1, we reproduce Figure 3 from the paper (adding one additional panel), which will help us

Date Printed. April 17, 2022.

¹Here $v \mapsto x_v$ is a given measurable function.

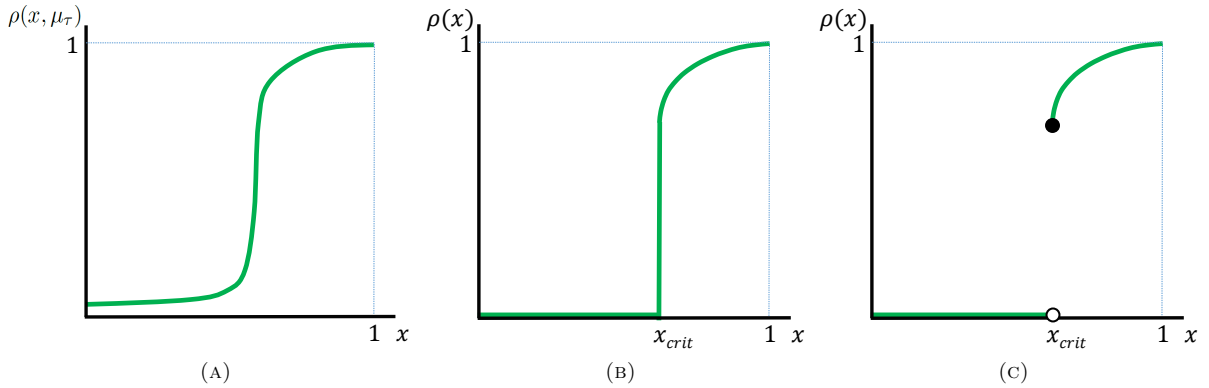


FIGURE 1. Panel (A) shows how reliability varies with the investment level x when supply chains have a finite depth. Panel (B) shows the limit correspondence of the relationships between reliability r and the investment level x as the depth gets large. Panel (C) shows how reliability varies with the investment level x for infinite depth.

describe what we are doing. First, we define a correspondence that will be central to our analysis. It is a limit of the functions $x \mapsto \rho(x, \mu_\tau)$, in the sense that their graphs converge to the graph of the correspondence $\rho(x)$ as $\tau \rightarrow \infty$ (see panels (A) and (B)). More formally:

DEFINITION SA1. Let $\rho : [0, 1] \rightrightarrows [0, 1]$ be a correspondence such that²

- (i) for any $r \in \rho(x)$, there is a sequence $\{x_\tau\}_{\tau=1}^\infty \rightarrow x$ such that $\lim_{\tau \rightarrow \infty} \rho(x_\tau, \mu_\tau) = r$;
- (ii) for any sequence $\{x_\tau\}_{\tau=1}^\infty \rightarrow x$ we have $\lim_{\tau \rightarrow \infty} \rho(x_\tau, \mu_\tau) \in \rho(x)$.

We will show that this limit correspondence ρ exists and is uniquely defined, and that it has the shape sketched in panel (B): zero until some point, with a vertical rise up to a certain critical level of reliability, followed by a concave ascent to reliability 1 at $x = 1$. With these results in hand, this subsection will culminate in the proofs of Proposition 1 in the paper and Proposition 2 in the paper.

Our main result on the reliability correspondence is:

PROPOSITION SA1 (Shape of the limit reliability correspondence ρ). Let the complexity of production be $m \geq 2$ and the number of potential suppliers for each input be $n \geq 2$. Then there is a unique ρ satisfying Definition SA1, and it has the following properties. There exists an $x_{\text{crit}} \in (0, 1)$ such that

- (i) $\rho(x)$ is single-valued for all $x \neq x_{\text{crit}}$;
- (ii) $\rho(x) = 0$ for all $x < x_{\text{crit}}$;
- (iii) there is a value $0 < \bar{r}_{\text{crit}} < 1$ such that $\rho(x_{\text{crit}}) = [0, \bar{r}_{\text{crit}}]$;
- (iv) $\rho(x)$ is strictly increasing in x for all $x > x_{\text{crit}}$;
- (v) $\lim_{x \downarrow x_{\text{crit}}} \rho'(x) = \infty$.

Sections SA2.1.1 and SA2.1.2 are devoted to proving this result. A closely related (discontinuous) function, sketched in panel (c), will play a role in this analysis, as we describe next. Once this is done, we will deduce Propositions 1 and 2 (in the paper).

SA2.1.1. The reliability function with infinite depth: A tool to study ρ . In order to prove Proposition SA1 it is helpful to define and characterize some properties of a certain function $\hat{\rho} : [0, 1] \rightarrow [0, 1]$, sketched in Figure 1(C). It will turn out that for all $x \neq x_{\text{crit}}$, $\rho(x) = \hat{\rho}(x)$. We define $\hat{\rho}(x)$ as the largest r solving the following equation

$$\hat{r} = (1 - (1 - x\hat{r})^n)^m. \quad (\text{SA-1})$$

(For intuition, note that this is equation (2) in the paper, which relates reliability at depths d and $d - 1$, but here with the same reliability r on both sides. This is an intuitive condition “at depth infinity” as reliability is not decreased by adding one layer of depth.)

²Equivalently, we can define ρ by saying that it is a correspondence such that the graphs of the functions $\rho(\cdot, \mu_\tau) : [0, 1] \rightarrow [0, 1]$ converge to the graph of ρ (in the Hausdorff set-distance metric).

Lemma SA1, which is proved in Section SA4.1, identifies several key properties of $\hat{\rho}(x)$.

LEMMA SA1. Suppose the complexity of the supply network is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (\text{SA-2})$$

Then there are values $x_{\text{crit}}, \bar{r}_{\text{crit}} \in (0, 1]$ such that:

- (i) $\hat{\rho}(x) = 0$ for all $x < x_{\text{crit}}$;
- (ii) $\hat{\rho}$ has a (unique) point of discontinuity at x_{crit} ;
- (iii) $\hat{\rho}$ is strictly increasing for $x \geq x_{\text{crit}}$;
- (iv) the inverse of $\hat{\rho}$ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[\bar{r}_{\text{crit}}, 1]$, where $\bar{r}_{\text{crit}} = \hat{\rho}(x_{\text{crit}})$;
- (v) χ is positive and quasiconvex on the domain $(0, 1]$;
- (vi) $\chi'(\bar{r}_{\text{crit}}) = 0$.

The proof of Lemma SA1 is in Section SA4.1. While the manipulations to establish these properties are a bit involved, they amount to studying the function $\hat{\rho}$ and the pseudo-inverse of it that we have defined, χ , using calculus. Figure 2(a) depicts χ as a function of r .³

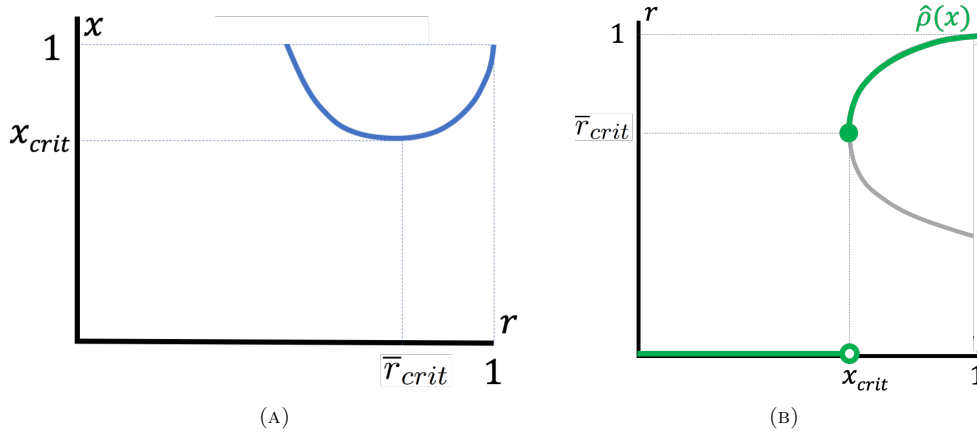


FIGURE 2. Panel (A) plots the function χ as r varies, and then in Panel (B) we show how switching the axes and taking the largest r value on the graph (corresponding to the largest solution of equation (SA-1)) generates $\hat{\rho}(x)$.

Recall the functions $\tilde{\rho}(x, d)$: consider a depth- d tree where each firm in each tier requires m kinds of inputs and has n potential suppliers of each input. We denote by $\tilde{\rho}(x, d)$ the probability of successful production at the most-downstream node of a depth- d tree with these properties. This is defined as

$$\tilde{\rho}(x, d) = (1 - (1 - x\tilde{\rho}(x, d-1))^n)^m$$

with $\tilde{\rho}(x, 0) = 1$, since the most-upstream tier nodes obtain their inputs without the possibility of disruption.

It will be useful throughout that $\tilde{\rho}(x, d)$ and its derivative converge uniformly to the function $\hat{\rho}(x)$ everywhere except near x_{crit} . Indeed, this will directly imply that reliability curve and the marginal returns to investment in our model also converge (at all points but x_{crit}) to their corresponding values in the “infinite-depth” model that defines $\hat{\rho}$. The following lemma formalizes these statements.

LEMMA SA2.

- (i) For all $d \geq 1$, the function $x \mapsto \tilde{\rho}(x, d)$ defined for $x \in (0, 1)$ is strictly increasing and infinitely differentiable.

³Since by definition the *largest* r satisfying (SA-1) is the one that determines $\hat{\rho}(x)$, it follows that the increasing part of the function, where $r \in [\bar{r}_{\text{crit}}, 1]$ is the part relevant for determining equilibrium reliability—see Figure 2(b), where the light gray branch is not part of $\hat{\rho}$.

- (ii) On any compact set excluding x_{crit} , the sequence $(\tilde{\rho}(x, d))_{d=1}^{\infty}$ converges uniformly to $\hat{\rho}(x)$.
- (iii) On any compact set excluding x_{crit} , the sequence $(\rho(x, \mu_{\tau}))_{\tau=1}^{\infty}$ converges uniformly to $\hat{\rho}(x)$.

Proof. The sequence $(\tilde{\rho}(\cdot, d))_{d=1}^{\infty}$ is a monotone sequence of increasing, infinitely differentiable functions,⁴ converging pointwise to $\hat{\rho}$. We know that $\hat{\rho}$ is continuous on any compact set excluding x_{crit} . Therefore, by Dini's theorem, the functions $\tilde{\rho}(\cdot, d)$ converge uniformly to $\hat{\rho}$.

Note the functions $\rho(\cdot, \mu_{\tau})$ are strictly increasing and infinitely differentiable, since they are the averages of such functions. Moreover, they clearly converge pointwise to $\hat{\rho}(x)$: since all the $\tilde{\rho}(\cdot, d)$ are uniformly bounded, the vanishing probability mass on low- d realizations makes only a negligible contribution to the value of $\rho(x, \mu_{\tau})$ for large τ . From this the conclusions of Lemma SA2 apply equally well to the sequence of functions $(\rho(\cdot, \mu_{\tau}))_{\tau=1}^{\infty}$. \square

SA2.1.2. *Proof of Proposition SA1.* We will now establish a close relationship between $\hat{\rho}$ and our correspondence of interest, ρ , which will allow us to use Lemmas SA1 and SA2 to prove Proposition SA1.

Define $\rho(x) = \{\hat{\rho}(x)\}$ for $x \neq x_{\text{crit}}$. By Lemma SA2(iii) $\rho(x, \mu_{\tau}) \rightarrow \hat{\rho}(x)$ as $\tau \rightarrow \infty$ for any $x \neq x_{\text{crit}}$. Thus Part (ii) of Definition SA1 is satisfied. Since $\rho(x)$ is single-valued for $x \neq x_{\text{crit}}$, Part (i) of Definition SA1 also holds. Lemma SA1 then implies points (i), (ii), (iv) and (v) of Proposition SA1.

Next, define $\rho(x_{\text{crit}}) = [0, \bar{r}_{\text{crit}}]$. We will show that the remaining conditions of Definition SA1 (those pertaining to x_{crit}) also hold. First, we will check Part (i) of Definition SA1 for $r \in \rho(x_{\text{crit}})$. For any $r \in [0, \bar{r}_{\text{crit}}]$, we can construct a sequence $\{x_{\tau}\}_{\tau=1}^{\infty} \rightarrow x_{\text{crit}}$ such that $\lim_{\tau \rightarrow \infty} \rho(x_{\tau}, \mu_{\tau}) = r$. To see this, simply note that for any $r \in (0, \bar{r}_{\text{crit}})$ and any τ , there is an x_{τ} such that $\rho(x_{\tau}, \mu_{\tau}) = r$ since $\rho(x, \mu_{\tau})$ is a continuous and increasing function of x whose image is $[0, 1]$. Moreover, since for any $x < x_{\text{crit}}$, $\lim_{\tau \rightarrow \infty} \rho(x, \mu_{\tau}) = 0$ and for any $x > x_{\text{crit}}$ there exists $\epsilon > 0$ such that $\lim_{\tau \rightarrow \infty} \rho(x, \mu_{\tau}) = \bar{r}_{\text{crit}} + \epsilon$, it follows that these x_{τ} must indeed converge to x_{crit} . In the case of $r = 0$, there exists a sequence $\{x_{\tau}\}_{\tau=1}^{\infty} \rightarrow x_{\text{crit}}$ such that $\rho(x_{\tau}, \mu_{\tau}) \downarrow 0$ since for any $x < x_{\text{crit}}$, $\rho(x, \mu_{\tau}) \downarrow 0$.

Finally, we show that every sequence $\{x_{\tau}\}_{\tau=1}^{\infty} \rightarrow x_{\text{crit}}$ satisfies $\lim_{\tau \rightarrow \infty} \rho(x_{\tau}, \mu_{\tau}) \in [0, \bar{r}_{\text{crit}}]$ (Part (ii) of Definition SA1). This amounts to showing that the limit is at most \bar{r}_{crit} . Suppose otherwise, that such a limit is $r' > \bar{r}_{\text{crit}}$. Let x' be such that $\hat{\rho}(x') = r'$, which exists by Lemma SA1. Note for all $x_{\tau} < x'$, we have $\hat{\rho}(x_{\tau}) < r' - \epsilon$ for some positive ϵ , and so $\rho(x_{\tau}, \mu_{\tau}) < r'$ for sufficiently large τ ; this is a contradiction to the hypothesis about the limit being r' .

These claims together establish the remaining content of the claim that the ρ we have defined is the limit satisfying Definition SA1. That implies part (iii) of Proposition SA1. \square

SA2.1.3. *Proof of Proposition 1 (Discontinuity in reliability) using the limit reliability correspondence.* We can now use the results established to prove the proposition about the precipice.

Part (i): The fact that $\rho(x, \mu_{\tau}) \rightarrow 0$ for $x < x_{\text{crit}}$ follows from Definition SA1 and Proposition SA1(ii).

Part (ii): From Proposition SA1(iii) and (iv), it follows that $\rho(x) > \bar{r}_{\text{crit}}$ for any $x > x_{\text{crit}}$ and thus, for τ large enough, $\rho(x, \mu_{\tau}) > \bar{r}_{\text{crit}} > 0$. \square

SA2.2. **Proof of Proposition 2 (Social planner's solution).** Now we can use the properties of the reliability curve to prove our result on the social planner's solution. Recall that $x^{\text{SP}}(\kappa, \mu_{\tau})$ is the set of all values of x maximizing the planner's objective.

First consider the following "limit" planner's problem, defined for $x \in [0, 1]$:

$$\max \left[h(r) - \frac{1}{\kappa} c_P(x) \right] \text{ subject to } r \in \rho(x). \quad (\text{SA-3})$$

We claim that, if $\delta > 0$ is chosen small enough, then any solution has $x \notin [x_{\text{crit}} - \delta, x_{\text{crit}} + \delta]$. To rule out solutions with $x \in [x_{\text{crit}} - \delta, x_{\text{crit}}]$, note that by Property B and continuity of c_P that the cost $c_P(x)$ is positive in that interval (if δ is fixed at a small enough value) while $\rho(x)$ is zero by Proposition 1 in the paper, so that $x = r = 0$ does better. To rule out solutions with $x \in [x_{\text{crit}}, x_{\text{crit}} + \delta]$, first observe that for $x = x_{\text{crit}}$, the only r that can be a solution is $r = r_{\text{crit}}$, so we may assume the solution lies on

⁴From eq. (2) in the paper, it is clear that increasing d decreases $\tilde{\rho}(x, d)$, while increasing x increases it. Differentiability is straightforward from the iterative definition.

the graph of $\widehat{\rho}$. Next, note that if δ is chosen small enough, then for any κ , we have $\widehat{\rho}'(x) > \frac{1}{\kappa}c_P(x)$ in the interval $[x_{\text{crit}}, x_{\text{crit}} + \delta]$ since $\widehat{\rho}'(x)$ tends to $+\infty$ as $x \downarrow x_{\text{crit}}$ by Lemma SA2.

Let $\kappa_{\text{crit}} > 0$ be the minimum value of κ such that that (SA-3) has a solution with $r > 0$. This κ_{crit} exists because for large enough κ we have that the maximand is positive at $x = x_{\text{crit}}$, and so there must be a solution with positive r ; similarly, κ_{crit} is positive since for small κ , the maximand is negative for all $x \geq x_{\text{crit}}$.⁵ By the previous paragraph, any solution at $\kappa = \kappa_{\text{crit}}$ satisfies $x > x_{\text{crit}}$. Since in the domain $x > x_{\text{crit}}$, we have that $\rho(x)$ is concave while c_P is convex, for $\kappa \geq \kappa_{\text{crit}}$ there is a unique, strictly positive solution $(x(\kappa), r(\kappa))$ of (SA-3) and both x and r are increasing in κ . For $\kappa < \kappa_{\text{crit}}$, we have that $\rho(x) = 0$ at the optimum and hence cost is zero (otherwise $r = x = 0$ would do better). Thus, by what we have said above,

- (i) for all $\kappa < \kappa_{\text{crit}}$, all solutions of (SA-3) have $x \leq x_{\text{crit}} - \delta$, have cost equal to 0, and yield reliability 0;
- (ii) for all $\kappa > \kappa_{\text{crit}}$, all solutions of (SA-3) have $x \geq x_{\text{crit}} + \delta$ and yield reliability $r \geq r_{\text{crit}}$;
- (iii) for $\kappa = \kappa_{\text{crit}}$, all solutions of (SA-3) are outside the interval $[x_{\text{crit}} - \delta, x_{\text{crit}} + \delta]$ and reliability is either 0 or strictly above r_{crit} .

Returning to the main model rather than the limit case above: for any κ and τ , consider the following rewriting of the planner's problem:

$$\max \left[h(r) - \frac{1}{\kappa}c_P(x) \right] \text{ subject to } r = \rho(x; \mu_\tau). \quad (\text{SA-4})$$

For any κ , it is straightforward to deduce from the definition of the correspondence ρ that a sequence of solutions $(x(\tau), \rho(x; \mu_\tau))$ of (SA-4) must converge to a solution of (SA-3) as $\tau \rightarrow \infty$.⁶ Thus, choosing ϵ small enough and using the convergence of $\rho(x; \mu_\tau)$ to $\rho(x)$ per Definition SA1 (along with the properties of ρ established in Proposition 1 in the paper), the conclusions of the present proposition follow. \square

SA2.3. Best responses and equilibrium behavior: Overview and preliminaries. Our goal now is to work toward Theorem 1 in the paper on the ordered regimes and the properties of equilibrium.

For notational convenience, when analyzing equilibria, we will multiply through all firms' profit functions by κ , so that a firm's cost is $c(x_{if} - \underline{x})$ and its gross profit is $\kappa g(r)P(x_{if}; r)$. Note this does not change any best responses. We maintain this convention throughout the remainder of the proofs, and in the Supplementary Appendix.

In Figure 3, we reproduce part of Figure 6 from the paper, which will guide our analysis. Before beginning, we outline the plan. At a high level, it amounts to bringing the best-response curve into the picture along with the precipice graph, and ultimately formalizing the graphical intuitions in Figure 3 concerning what happens to the best-response curve as κ moves around. Throughout, a recurring strategy is to prove a result for the game "at the $\tau = \infty$ " limit (in a sense we make precise), and then extend the conclusion to large values of τ , where most of the mass is on deep supply trees.

To study this $\tau = \infty$ limit, we establish some notation. First, define

$$\widehat{P}(x_{if}; r) = (1 - (1 - x_{if}r)^n)^m. \quad (\text{SA-5})$$

This is the probability with which if is able to produce, as a function of its investment choice x_{if} given that all its suppliers have reliability r . This is simpler than our finite- τ problem because all firms are exactly symmetric (which is only approximately true when τ is large but finite).

Define

$$MB(x_{if}; r, \kappa) = \kappa g(r) \frac{\partial \widehat{P}(x_{if}; r)}{\partial x_{if}} \quad (\text{SA-6})$$

$$MC(x_{if}) = c'(x_{if} - \underline{x}). \quad (\text{SA-7})$$

These are the marginal benefit and marginal cost, respectively, to a firm of investing in relationship strength for the supply network "at the limit."

⁵Since $c_P(x_{\text{crit}}) > 0$ and c_P is convex, $c_P(x) > 0$ for all $x \geq x_{\text{crit}}$. Thus by making κ small enough, the costs exceed the bounded benefits. We can say "minimum" rather than "infimum" in the definition of κ_{crit} because the correspondence ρ is continuous, as is c_P .

⁶This is a version of the Theorem of the Maximum on upper hemicontinuity of optimization.

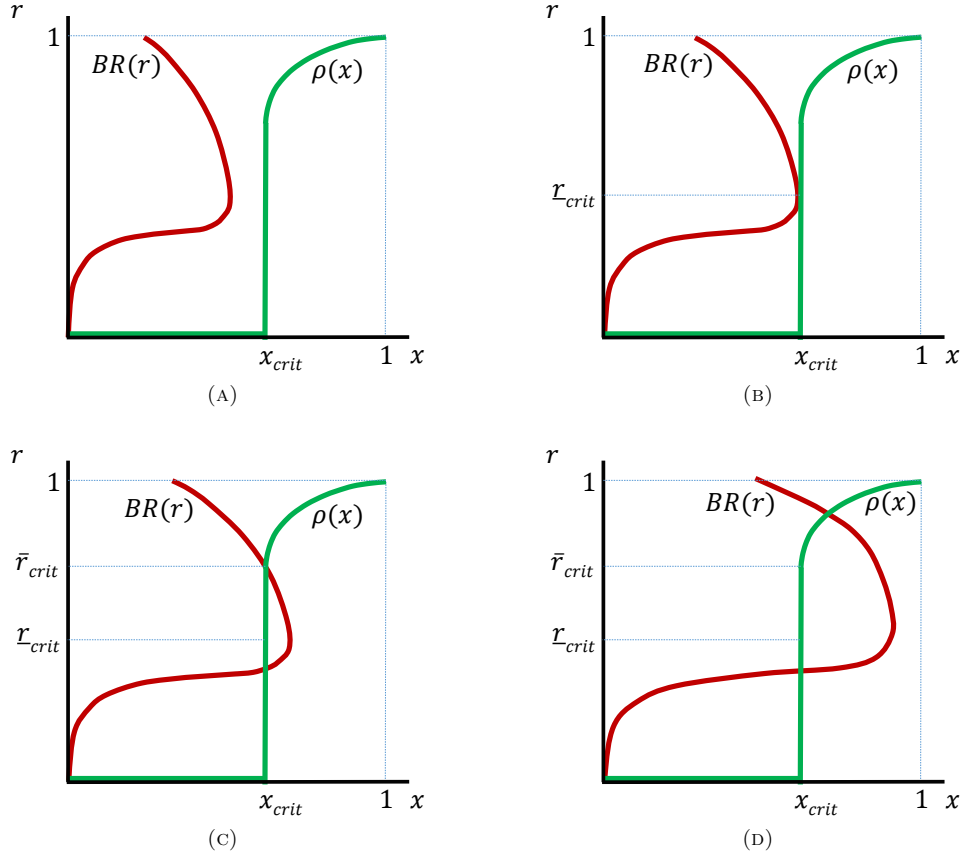


FIGURE 3. Panel (A) shows an equilibrium for $\kappa < \bar{\kappa}$. Panel (B) shows an equilibrium with $\kappa = \bar{\kappa}$. Panel (C) shows an equilibrium with $\kappa = \bar{\kappa}$. Panel (D) shows an equilibrium with $\kappa > \bar{\kappa}$. Panel (E) plots how equilibrium reliability varies with κ . Panel (F) shows reliability following an arbitrarily small negative shock to institutional quality \underline{x} as κ varies.

A first step in the analysis is making best responses at $\tau = \infty$ tractable by expressing them as the solution to a first-order condition equating marginal benefits and marginal costs, which is unique under suitable assumptions. This is done in Section SA2.3.1. Second, we show in Section SA2.3.2 that if we are interested in equilibria that are undominated in terms of social surplus, we may focus on the higher points of intersection in Figure 3. Third, we show in Section SA2.3.3 that in the limit model, there is a unique intersection between the two curves above reliability \bar{r}_{crit} , as in Figure 3(d), with the best-response curve sloping down in r , which makes the comparative statics work as in the sketches.

Together, these ingredients set up the proof of Theorem 1 from the paper in Section SA2.4, allowing us to formalize the graphical intuition. The remaining subsections carry out this plan.

SA2.3.1. *Proof of Lemma 1 (Sufficient condition for unique interior local maximum of the firm's objective).* We first study the firm's problem "at $\tau = \infty$." Afterward, we establish that the same uniqueness property holds for large, finite τ .

For the extended domain $x_{if} \in [0, 1/r]$, we define

$$Q(x_{ik}; r) := \frac{\partial}{\partial x_{if}} \hat{P}(x_{if}; r), \quad (\text{SA-8})$$

which can be calculated to be $Q(x_{ik}; r) = mn(1 - (1 - x_{if}r)^n)^{m-1}(1 - x_{if}r)^{n-1}r$.

We will need two steps to prove Lemma 1. The first step consists of establishing Lemma SA3 on the basic shape of $Q(x_{if}; r)$. Figure 4 illustrates the shape of $Q(x_{if}; r)$ implied by Lemma SA3.

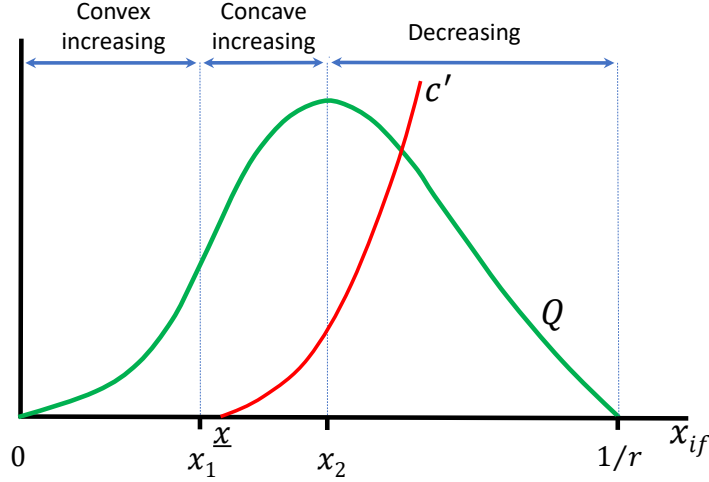


FIGURE 4. The shape of $Q(x_{if}, x)$ in green and $c'(x_{if} - x)$ in red. There can be only one intersection between the two curves, which corresponds to the maximizer of $\Pi(x_{if}; r)$. (Here we normalized $\kappa g(r) = 1$ for simplicity, but the illustration remains valid if the green curve is scaled.)

LEMMA SA3. Fix any $m \geq 2$, $n \geq 2$, and $r \geq r_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n , and x) such $0 \leq x_1 < x_2 < 1/r$ and so that:

0. $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$;
1. $Q(x_{if}; r)$ is increasing and convex in x_{if} on the interval $[0, x_1]$;
2. $Q(x_{if}; r)$ is increasing and concave in x_{if} on the interval $(x_1, x_2]$;
3. $Q(x_{if}; r)$ is decreasing in x_{if} on the interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

The proof of Lemma SA3 is in Section SA4.2.

We now complete the proof of Lemma 1 by setting $\hat{x} = x_1$. By Lemma SA3(4), the interval $(\hat{x}, x_{\text{crit}})$ is non-empty. Thus we just need to show that Assumption 1 from the paper (the sufficiency of the first-order condition for interior optima) is satisfied when $\underline{x} \in (\hat{x}, x_{\text{crit}})$. Note that by Property B' in the paper, we have that $c'(0) = 0$ and c' is weakly increasing and weakly convex otherwise. Since, by Lemma SA3, $P'(x_{if}; x)$ is first concave and increasing (possibly for the empty interval) and then decreasing (possibly for the empty interval) over the range $x_{if} \in [\underline{x}, 1]$, it follows that there is at most a single intersection between the curves $P'(x_{if}; x)$ and $c'(x_{if} - \underline{x})$. This intersection corresponds to the first-order condition $Q(x_{if}; r) = c'(x_{if} - \underline{x})$, yielding the unique maximizer of $\Pi(x_{if}; x)$, as illustrated in Figure 4. If such an intersection not exist, $y_{if} = 0$ is a local and global maximizer of the profit function.

Now we study the quantities introduced above, but for finite a τ and show that when τ is large enough, we still obtain a unique solution to the firm's problem.

Recall that

$$P(x_{if}; x, \mu_\tau) = \mu_\tau(0) + \sum_d \mu_\tau(d) (1 - (1 - x_{if} \tilde{\rho}(x, d-1))^n)^m.$$

Define

$$Q(x_{if}; x, \mu_\tau) = \frac{\partial}{\partial x_{if}} P'(x_{if}; x, \mu_\tau) = \sum_d \mu_\tau(d) m n (1 - (1 - x_{if} \tilde{\rho}(x, d-1))^n)^{m-1} (1 - x_{if} \tilde{\rho}(x, d-1))^{n-1} \tilde{\rho}(x, d-1).$$

Recall that for all $x \neq x_{\text{crit}}$ we have

$$\tilde{\rho}(x, d) \rightarrow_d \hat{\rho}(x) \text{ and } \rho(x, \mu_\tau) \rightarrow_\tau \hat{\rho}(x).$$

It follows from this and the expression for $Q(x_{if}; x, \mu_\tau)$ above that

$$P(x_{if}; x, \mu_\tau) \rightarrow_\tau \hat{P}(x_{if}; \hat{\rho}(x)) \text{ and } Q(x_{if}; x, \mu_\tau) \rightarrow_\tau Q(x_{if}; \hat{\rho}(x)).$$

It follows that marginal costs and marginal benefits in the $\tau \rightarrow \infty$ limit are arbitrarily close to the functions studied above. We then conclude that for τ large enough, there will also be at most a single

intersection between the curves $Q(x_{if}; x, \mu_\tau)$ and $c'(x_{if} - \underline{x})$ and thus the results stated in the limit also hold for τ large enough. \square

SA2.3.2. A lemma on symmetric undominated equilibria. This subsection establishes that the symmetric equilibrium maximizing social surplus is the one with greatest relationship strengths x and highest reliability. In other words, it is the higher intersection in Figure 3 when there are several. Normalizing $\kappa = 1$ without loss of generality, recall our notation that at a symmetric equilibrium with reliability r , gross output is $h(r)$ and gross profits are $g(r)$.

LEMMA SA4. Consider two symmetric equilibria with relationship strengths $x_1 < x_2$ and reliabilities $r_1 < r_2$. Then $h(r_2) - c_P(x_2) > h(r_1) - c_P(x_1)$. That is, the higher-investment equilibrium has the greater net social surplus.

Proof. Let $V(x) = h(\rho(x, \mu)) - c_P(x)$. We will first show that V is increasing at $x = x_2$, and then use this to deduce the conclusion.

Consider a symmetric outcome with firm investments $(x_v)_{v \in \mathcal{V}}$ and a reliability of r . Note that the gross profit per functional firm is $g(r)$ so that the gross profit integrated across all firms (functional or nonfunctional) is $rg(r)$.

We will write net social surplus as follows,

$$\begin{aligned} V &= h(r) - \int_{\mathcal{V}} c(x_v) dv \\ &= h(r) - \int_{\mathcal{V}} \Pi_v(x_v; r) dv + \int_{\mathcal{V}} \Pi_v(x_v; r) dv - \int_{\mathcal{V}} c(x_v) dv \\ &= [h(r) - rg(r)] + \underbrace{\int_{\mathcal{V}} [\Pi_v(x_v; r) - c(x_v)] dv}_{\text{Net producer surplus}}. \end{aligned}$$

We want to consider a simultaneous and identical increase in the individual level investments x_v . For this purpose, a change of variable is helpful and we let $x_v = \xi_v + t$. Note that r is a function of the $(x_v)_v$ and thus of t .

We want to totally differentiate V in t and show the derivative is positive.

$$\begin{aligned} \frac{d}{dt} V &= \frac{d}{dt} [h(r) - rg(r)] + \frac{\partial}{\partial r} \int_{\mathcal{V}} [\Pi_v(x_v; r) - c(x_v)] dv \frac{dr}{dt} + \int_{\mathcal{V}} \frac{\partial [\Pi_v(x_v; r) - c(x_v)]}{\partial x_v} \frac{dx_v}{dt} dv \\ &= \frac{d}{dt} [h(r) - rg(r)] + \frac{\partial}{\partial r} \int_{\mathcal{V}} [\Pi_v(x_v; r) - c(x_v)] dv \frac{dr}{dt} \\ &= \frac{d}{dt} [h(r) - rg(r)] + \frac{\partial}{\partial r} rg(r) \frac{dr}{dt} \\ &= \frac{d}{dt} h(r) \\ &= h'(\rho(x, \mu)) \left[\frac{d}{dx} \rho(x, \mu) \right]. \end{aligned}$$

The second equality holds because we are in equilibrium and firms are optimizing. The third equality because $c(x_v)$ does not vary in r and because an alternative expression for gross producer surplus is $rg(r)$. The fourth equality cancels the $\frac{d}{dt} rg(r)$ terms and the last equality follows from outcome being symmetric such that $x_v = x$ for all v .

As both factors in the last expression in the above sequence of equalities are clearly positive, so V is increasing in t .

Note that for $x \geq x_{\text{crit}}$, we have that h is concave by Property A and increasing in x while $c_P(x)$ is convex, so

$$h(\rho(x, \mu)) - c_P(x)$$

is single-peaked. The above argument implies both equilibria are socially inefficient, and thus social surplus is increasing in x at both of them. This combined with single-peakedness shows that the one with higher relationship strength is more efficient. \square

SA2.3.3. *A limit uniqueness result for equilibria above \bar{r}_{crit} .* In this section we prove an important lemma that will be used to ensure that the relationship strengths and reliability achieved at symmetric undominated equilibria are uniquely determined. As before, we study the “ $\tau = \infty$ ” model as a stepping stone to our large- τ results. Visually, the lemma establishes that there is at most one intersection in Figure 3 that lies above \bar{r}_{crit} .

To motivate the statement, consider the question: when can there be an equilibrium outcome with $r \geq \bar{r}_{crit}$ for the limit payoffs at $\tau = \infty$? Recall the definitions of marginal benefits and marginal costs from (SA-6) and (SA-7) above. Under Assumption 1 from the paper, an interior firm optimum is characterized by the first order condition (recall equations (SA-6) and (SA-7) for the definitions)

$$MB(x; r, \kappa) = MC(x), \quad (\text{OI})$$

which we have labeled OI, for optimal investment. In addition, an outcome of the $\tau = \infty$ model also satisfies $r = \hat{\rho}(x)$ for $r \geq \bar{r}_{crit}$: the reliability is the one induced by firms’ choices of relationship strengths. Recalling the definition of χ from Lemma SA1, this entails $x = \chi(r)$. Then we have the equation

$$MB(\chi(r); r, \kappa) = MC(\chi(r)).$$

We will show that (for some $\epsilon > 0$) there is at most one solution (x, r) satisfying $r \in [\bar{r}_{crit} - \epsilon, 1]$ and $x = \chi(r)$ simultaneously. This implies that there is only one solution to the first-order conditions above \bar{r}_{crit} , and the fact that the necessary condition cannot hold even for slightly lower values of \bar{r}_{crit} will be technically useful later.

LEMMA SA5. Fix any $n \geq 2$ and $m \geq 3$. There exists $\epsilon > 0$ such that the equation $MB(\chi(r); r, \kappa) = MC(\chi(r))$ has at most one solution r^* in the range $[\bar{r}_{crit} - \epsilon, 1]$.

Proof. In the following argument, we defer technical steps to lemmas, which are proved in the Supplementary Appendix.

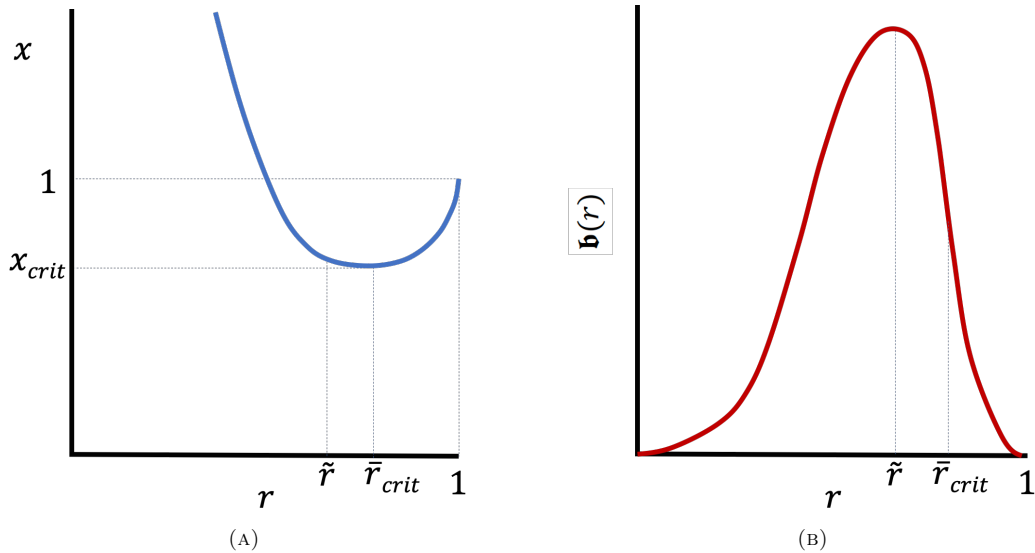


FIGURE 5. Panel (A) shows the relationship between r and x implied by physical consistency. Panel (B) plots the function $\mathbf{b}(r)$ discussed in the proof.

Define $\mathbf{b}(r) = MB(\chi(r); r, \kappa)$ and $\mathbf{c}(r) = MC(\chi(r))$, so that we can simply study the equation

$$\mathbf{b}(r) = \mathbf{c}(r). \quad (\text{SA-9})$$

We can calculate

$$MB(x_{if}; r, \kappa) = \kappa g(r) r n (1 - x_{if} r)^{n-1} m (1 - (1 - x_{if} r)^n)^{m-1}. \quad (\text{SA-10})$$

By plugging in $x = \chi(r)$ into (SA-10), we may rewrite the equation of interest as

$$\underbrace{\kappa g(r) m n r^{2 - \frac{1}{m}} \left(1 - r^{1/m}\right)^{1 - \frac{1}{n}}}_{\mathbf{b}(r)} = c' \left(\underbrace{\frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}}_x - \underline{x} \right). \quad (\text{SA-11})$$

We wish to show that there is a range $[\bar{r}_{\text{crit}} - \epsilon, 1]$ for which there is at most one solution to equation (SA-9).⁷ The right-hand side of the equation is increasing in r for $r \in [\bar{r}_{\text{crit}}, 1]$.⁸ If we could establish that the left-hand side, which we call $\mathbf{b}(r)$, is decreasing in r , it would follow that there is at most a unique r solving (SA-11) for $r \in [\bar{r}_{\text{crit}}, 1]$. Moreover, by the continuity of $\mathbf{b}(r)$ and $c(r)$, it would follow immediately that for some small $\epsilon > 0$, there is also at most a single solution over the range $[\bar{r}_{\text{crit}} - \epsilon, 1]$. Unfortunately, $\mathbf{b}(r)$ is not decreasing in r . However, we can show that it *is* decreasing in r for $r \geq \bar{r}_{\text{crit}}$, which is sufficient.⁹ Panel (B) of Figure 5 gives a representative depiction of $\mathbf{b}(r)$, reflecting that it is decreasing to the right of \tilde{r} (which is itself to the left of \bar{r}_{crit}).

The following lemma will, once established, complete the proof.

LEMMA SA6. \mathbf{b} is strictly decreasing on the domain $[\bar{r}_{\text{crit}}, 1]$.

To prove Lemma SA6 we write $\mathbf{b}(r)$ as a product of two pieces, $\alpha(r) := \kappa g(r)$ and

$$\beta(r) := m n r^{2 - \frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1 - \frac{1}{n}}.$$

Note that the function $\beta(r)$ is positive for $r \in (0, 1)$. We will show that it is also strictly decreasing on $[\bar{r}_{\text{crit}}, 1]$. By assumption, $g(r)$ is positive and strictly decreasing in its argument, so $\alpha(r)$ is also positive and decreasing in r . Thus, because \mathbf{b} is the product of two positive, strictly decreasing functions on $[\bar{r}_{\text{crit}}, 1]$, it is also strictly decreasing on $[\bar{r}_{\text{crit}}, 1]$. It remains only to establish that $\beta(r)$ is strictly decreasing on the relevant domain. Two additional lemmas are helpful.

LEMMA SA7. The function $\beta(r)$ is quasiconcave and has a maximum at $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

LEMMA SA8. For all $n \geq 2$ and $m \geq 3$, we have that $\hat{r} < r_{\text{crit}}$.

Lemmas SA7 and SA8 are proved in Sections SA4.3 and SA4.4. Together these show that $\beta(r)$ is strictly increasing and then strictly decreasing in r for $r \in (0, 1)$, with a turning point in the interval $(0, \bar{r}_{\text{crit}})$. Thus $\beta(r)$ is strictly decreasing on the domain $[\bar{r}_{\text{crit}}, 1]$, the final piece required to prove Lemma SA6.

Thus, by the continuity of $\mathbf{b}(r)$ and $c(r)$, it follows immediately that there exists $\epsilon > 0$ such that there is at most a single solution for $r \in [\bar{r}_{\text{crit}} - \epsilon, 1]$. This completes the proof of the lemma. \square

SA2.4. Proof of Theorem 1 (Classification of regimes as κ varies). Having developed the key machinery that we will use to analyze the comparative statics of equilibria, we can proceed to the main proof. We will sometimes take $\underline{x} = 0$ when this is immaterial to the arguments to keep notation uncluttered.

The proof relies mainly on analyzing the shape of the best-response correspondence: how it depends on κ and the reliability level r .

Denote by $BR(r, \kappa)$ the set of values x_{if} maximizing

$$\kappa g(r) \hat{P}(x_{if}; r) - c(x_{if} - \underline{x}). \quad (\text{SA-12})$$

Recalling the definition (SA-5), this has the interpretation of best-response relationship strengths in the infinite-depth model. Let $\bar{BR}(r, \kappa) = \max\{x : x \in BR(r, \kappa)\}$ be the maximal element of the set $BR(r, \kappa)$. Analogously, let $\bar{BR}(r, \kappa, \mu_\tau) = \max\{x : x \in BR(r, \kappa, \mu_\tau)\}$. This is the finite- τ analogue of the limit object we have just defined.

⁷We note that the functions on both sides of the equation are merely constructs for the proof. In particular, when we sign their derivatives, these derivatives do not have an obvious economic meaning.

⁸This follows directly: χ is increasing on that domain and c' is increasing by assumption. (see Panel (A) of Figure 5).

⁹We do this by showing that the global maximum of $\mathbf{b}(r)$ is achieved at a number \tilde{r} that we can prove is smaller than \bar{r}_{crit} .

Recall the optimal investment first-order condition (OI) in the limit model:

$$MB(x; r, \kappa) = MC(x), \quad (\text{OI})$$

where the definitions are in (SA-6) and (SA-7). We will be using it to study interior optima.

A first lemma in the proof of Theorem 1 is monotonicity of the best response curve in κ .

LEMMA SA9. $\overline{BR}(r, \kappa)$ is increasing in κ , and strictly so whenever $\overline{BR}(r, \kappa) > 0$.

Proof. Let $x_{if} = \overline{BR}(r, \kappa)$ and suppose OI holds at the optimum. Suppose κ increases slightly. Then, evaluated at x_{if} , the left-hand side of OI increases and the right-hand side does not change as it does not depend on κ . So marginal benefits exceed marginal costs at x_{if} . Now consider increasing x_{if} . By the assumption that $c'(x') \rightarrow \infty$ as $x' \rightarrow 1$ and the boundedness of marginal benefits, there is an $x' > x$ so that marginal benefits are once again equal to marginal costs. Since the original best-response gave a nonnegative payoff, so must x' (since marginal benefits exceed marginal costs in moving from x to x'). By Assumption 1 from the paper, the first-order condition is sufficient for a best-response.

On the other hand, if $\overline{BR}(r, \kappa) = 0$ is a strict best response, then by continuity of the value function, it remains so for a slightly higher κ . \square

We continue with two lemmas on the best response correspondence. Thereafter, we will prove the three claims of the theorem.

The first lemma concerns the local comparative statics of the intersection between the graph of $r \mapsto \overline{BR}(r, \kappa)$ and the graph of ρ .

LEMMA SA10. Take any κ_0 . Consider any (x_0, r_0) satisfying

$$x = \overline{BR}(r, \kappa) \text{ and } r \in \rho(x) \quad (\text{SA-13})$$

with $x_0 > x_{\text{crit}}$ at $\kappa = \kappa_0$. Then there is a neighborhood around κ_0 where $(X(\kappa), R(\kappa))$ is the unique pair satisfying (SA-13), and both X and R are strictly decreasing, continuous functions of κ .

Proof. Restrict attention to a neighborhood of (x_0, r_0) where $x > x_{\text{crit}}$ and $r > r_{\text{crit}}$. Here ρ is the same as $\hat{\rho}$. Note that in the notation of Lemma SA5, the conditions (SA-13) are equivalent to

$$\mathbf{b}(r) = \mathbf{c}(r) \text{ and } x = \chi(r).$$

The existence of continuous functions $(X(\kappa), R(\kappa))$ extending the solution locally follows by the implicit function theorem applied to this equation. The solution $R(\kappa)$ can be visualized as the horizontal coordinate where the decreasing function $\mathbf{b}(r)$ in Figure 5(b) intersects the increasing function $c'(\chi(r) - \underline{x})$. As we perturb κ upward, the curve $\mathbf{b}(r)$ moves up pointwise. Then $R(\kappa)$, the coordinate of the intersection, is increasing in κ . The statement about $X(\kappa)$ follows because X is an increasing function of $R(\kappa)$ in our neighborhood of interest. \square

The following lemma concerns the convergence of the maximal best-response function $\overline{BR}(r, \kappa, \mu_\tau)$ in the finite- τ model to the limit model.

LEMMA SA11. $\overline{BR}(r, \kappa, \mu_\tau)$ converges pointwise to $\overline{BR}(r, \kappa)$ for any r .

Proof of Lemma SA11. When it is nonzero, $\overline{BR}(r, \kappa, \mu_\tau) = \max\{x : MB(x; r, \kappa, \mu_\tau) = MC(x)\}$ where

$$MB(x; r, \kappa, \mu_\tau) = \kappa g(r) m n (1 - xr)^{n-1} (1 - (1 - xr)^n)^{m-1} r \sum_{d=1}^{\infty} \mu_\tau(d).$$

Suppose first (passing to a subsequence if necessary) we have a sequence where these conditions hold.

Note that, since μ_τ puts mass at least $1 - \frac{1}{\tau}$ on $[\tau, \infty)$, it follows that $\sum_{d=1}^{\infty} \mu_\tau(d) \rightarrow 1$ as $\tau \rightarrow \infty$.

It follows that, as $\tau \rightarrow \infty$, we have the following convergence uniformly:

$$MB(x; r, \kappa, \mu_\tau) \rightarrow \kappa g(r) m n (1 - xr)^{n-1} (1 - (1 - xr)^n)^{m-1} r = MB(x; r, \kappa)$$

and thus that

$$\overline{BR}(r, \kappa, \mu_\tau) \rightarrow \overline{BR}(r, \kappa)$$

for any r .

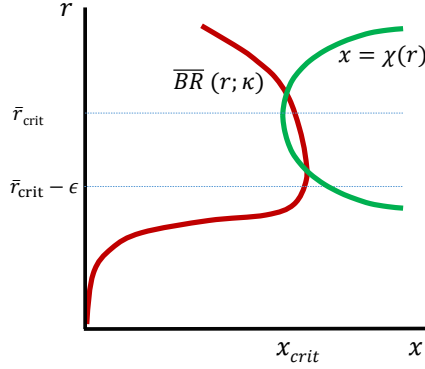


FIGURE 6. The curves $\overline{BR}(r, \kappa)$ and $x = \chi(r)$. If the two were tangent at $r = \bar{r}_{crit}$ at $\kappa = \underline{\kappa}$, then by perturbing to a slightly higher κ , we would obtain two intersections above $r = \bar{r}_{crit} - \epsilon$, contradicting Lemma SA5.

When $\overline{BR}(r, \kappa, \mu_\tau)$ is identically zero for each τ (again by passing to a subsequence if necessary), it follows from uniform convergence of the firms' value functions to their limit (SA-12) that investing zero strictly dominates any positive level in the limit problem as well. \square

We now define some key quantities that will play a role in our proof of the theorem. It is straightforward to see by inspection of the limit maximand (SA-12) and condition (OI) that for any $r \in (0, 1]$ and any $x_0 < 1$, if κ is large enough, then all elements of $BR(r, \kappa)$ satisfy $x > x_0$. In particular, we can guarantee that they all lie above x_{crit} . What we have said implies that there is an κ_1 such that for $\kappa > \kappa_1$ the graphs of $\overline{BR}(r, \kappa)$ and the graph of ρ (both viewed as sets of points (x, r)) intersect above x_{crit} . On the other hand, at sufficiently small κ , all elements of $BR(r, \kappa)$ are bounded by a small number. Let $\underline{\kappa} > 0$ be the smallest κ such the graph of $\overline{BR}(r, \kappa)$ intersects the graph of ρ at a nonzero value of x . What we have said implies that $\underline{\kappa}$ is a finite, nonzero number.

Consider the two graphs just mentioned at $\kappa = \underline{\kappa}$. Lemma SA10 implies that any intersection of the two graphs must be such that $x \leq x_{crit}$ (and thus $r \leq \bar{r}_{crit}$), since otherwise we could find an intersection at a smaller value of κ . Thus, $\underline{\kappa}$ is such that $\overline{BR}(r, \underline{\kappa})$ just touches the correspondence $\rho(x)$ at $x = x_{crit}$. Define \underline{r}_{crit} to be the point of intersection with the highest r , as depicted in Figure 3(c). By Assumption 1 in the paper, we may assume that locally, $\overline{BR}(r, \underline{\kappa})$ is exactly the set of solutions to (OI) as a function of r .¹⁰ If it were the case that $\underline{r}_{crit} = \bar{r}_{crit}$, then $r \mapsto \overline{BR}(r, \underline{\kappa})$ would be tangent to $r \mapsto \chi(r)$ at $r = \bar{r}_{crit}$, and by increasing κ slightly we would obtain two intersections between the graphs above $\bar{r}_{crit} - \epsilon$ a contradiction to Lemma SA5. Thus $\underline{r}_{crit} < \bar{r}_{crit}$.

We can now prove each part of Theorem 1.

Part (i): For any $\kappa < \underline{\kappa}$, Since $\overline{BR}(0, \kappa) = 0$, the only intersection between $\overline{BR}(r, \kappa)$ and $\rho(x)$ is at $(x, r) = (0, 0)$.

Note that for any τ , $\frac{\partial \rho(x, \mu_\tau)}{\partial x_{if}}|_{x_{if}=x=0} = 0$ and $\overline{BR}(0, \kappa) = 0$. Now, since we know $\rho(x, \mu_\tau)$ converges to $\rho(x)$ (Proposition SA1) and $\overline{BR}(r, \kappa, \mu_\tau)$ converges to $\overline{BR}(r, \kappa)$ (Lemma SA11), it follows that when $\kappa < \underline{\kappa}$, then there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, the curves $\overline{BR}(r, \kappa, \mu_\tau)$ and $\rho(x, \mu_\tau)$ intersect only at a point $(x, r) = (0, 0)$.

Part (ii): Now take any $\kappa \geq \underline{\kappa}$. Since $\overline{BR}(r, \kappa) \geq \overline{BR}(r, \underline{\kappa})$ for any r (with equality holding only when $\kappa = \underline{\kappa}$ or possibly when $\overline{BR}(r, \kappa) = 0$), it follows that there is at least one point of intersection between $\overline{BR}(r, \kappa)$ and $\rho(x)$. We select the one with the highest reliability r . If κ is close enough to $\underline{\kappa}$, then this point will be (x_{crit}, r) with $r \in (\underline{r}_{crit}, \bar{r}_{crit})$.

Since we know $\rho(x, \mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r, \kappa, \mu_\tau)$ converges to $\overline{BR}(r, \kappa)$, it then follows that for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, $\overline{BR}(r, \kappa, \mu_\tau)$ and $\rho(x, \mu_\tau)$ intersect at some point $x \in [x_{crit} - \epsilon, x_{crit} + \epsilon]$ and $r \in [\underline{r}_{crit} - \epsilon, \bar{r}_{crit} + \epsilon]$.

¹⁰If the best response is indifferent to zero investment, then this indifference can be broken by changing the cost function slightly at inframarginal costs without affecting the equilibrium.

As κ keeps increasing, $\overline{BR}(r, \kappa)$ also increases and thus the point of intersection with the highest reliability will reach $(x_{\text{crit}}, \bar{r}_{\text{crit}})$ when κ reaches some value $\bar{\kappa}$. Since we know $\rho(x, \mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r, \kappa, \mu_\tau)$ converges to $\overline{BR}(r, \kappa)$, by the same argument as before, it then follows that for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for $\tau > \underline{\tau}$, $\overline{BR}(r, \kappa, \mu_\tau)$ and $\rho(x, \mu_\tau)$ intersect at some point $x \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$ and $r \in [\bar{r}_{\text{crit}} - \epsilon, \bar{r}_{\text{crit}} + \epsilon]$.

Part (iii): Finally, as κ increases beyond $\bar{\kappa}$, $\overline{BR}(r, \kappa)$ keeps increasing and the point of intersection between $\overline{BR}(r, \kappa)$ and $\rho(x)$ with the highest reliability r will increase along the part of the $\rho(x)$ curve for which $x > x_{\text{crit}}$, by Lemma SA10. At any such point, we also have $r > \bar{r}_{\text{crit}}$.

Since $\rho(x, \mu_\tau)$ converges to $\rho(x)$ and $\overline{BR}(r, \kappa, \mu_\tau)$ converges to $\overline{BR}(r, \kappa)$, this also holds for any τ large enough. This completes the proof. \square

SA2.4.1. *Proof of Corollary 1 (Comparative statics in baseline institutional quality).* We first consider the limit model. Equilibria are given by the highest intersection of the reliability curve with the best response curve. The reliability curve is constant as both κ and \underline{x} change. The conditions for optimal investment that must be satisfied in the equilibrium $x^{*'} := x^*(\kappa', \underline{x})$ are that

$$MB(x^{*'}) = \frac{1}{\kappa'} c'(x^{*'} - \underline{x}).$$

It needs to be shown that there exists an $\underline{x}' \in (\underline{x}, x^{*'})$ such that

$$MB(x^{*'}) = \frac{1}{\kappa} c'(x^{*'} - \underline{x}').$$

Note that $c'(x^{*'} - \underline{x}')$ is continuous in \underline{x}' holding $x^{*'}$ fixed, with

$$\frac{1}{\kappa} c'(0) = 0 < \frac{1}{\kappa'} c'(x^{*'} - \underline{x}) = MB(x^{*'}) \quad \text{and} \quad \frac{1}{\kappa} c'(x^{*'} - \underline{x}) > \frac{1}{\kappa'} c'(x^{*'} - \underline{x}) = MB(x^{*'}).$$

The result then follows from the intermediate value theorem.

The extension to the limit as $\tau \rightarrow \infty$ is straightforward and analogous to other arguments. \square

SA2.4.2. *Proof of Proposition 3 (Equilibrium fragility).* When $\kappa \in [\underline{\kappa}, \bar{\kappa}]$:

From Theorem 1 in the paper, for any $\epsilon > 0$, there exists $\underline{\tau}$ such that for any $\tau > \underline{\tau}$, $x^*(\mu_\tau) \in [x_{\text{crit}} - \epsilon, x_{\text{crit}} + \epsilon]$. Thus for a shock of size 2ϵ , the relationship strength after the shock is $\underline{x} - \epsilon + \xi^*(\mu_\tau) \leq x_{\text{crit}} - \epsilon$.

From Proposition SA1, for any $\eta > 0$, there exists $\underline{\tau}'$ such that for all $\tau > \max\{\underline{\tau}, \underline{\tau}'\}$, $\rho(\underline{x} - \epsilon + y^*(\mu_\tau), \mu_\tau) < \eta$.

Thus, when $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ the equilibrium is fragile.

When $\kappa > \bar{\kappa}$:

From Theorem 1 in the paper, $x^*(\mu_\tau)$ converges to $x^* > x_{\text{crit}}$ as $\tau \rightarrow \infty$. Thus, for any $\epsilon > 0$, there may *not* exist a $\underline{\tau}$ such that $x^*(\mu_\tau) - \epsilon = \underline{x} - \epsilon + y^*(\mu_\tau) < x_{\text{crit}}$ for all $\tau > \underline{\tau}$ and from Proposition SA1, for any η , there may *not* exist a $\underline{\tau}'$ such that $\rho(\underline{x} - \epsilon + y^*(\mu_\tau), \mu_\tau) < \eta$ for all $\tau > \max\{\underline{\tau}, \underline{\tau}'\}$.

Thus, when $\kappa > \bar{\kappa}$ the equilibrium is robust. \square

SA2.4.3. *Proof of Proposition 4 (Fragility with partial knowledge of depth).* It is helpful to first define and analyze an adjusted baseline model, called the r_A -baseline model. This model is identical to the baseline model except that the reliability of depth 0 firms is replaced by the adjusted reliability $r_A < 1$.

Reliability at depth d in this adjusted baseline model is $\mathcal{R}_x^d(r_A)$ (with \mathcal{R}^d denoting applying the function d times). This is less than reliability in the baseline model at depth d , given by $\mathcal{R}_x^d(1)$ because the function \mathcal{R}_x is increasing. However, if $r_A \geq \bar{r}_{\text{crit}}$ from the baseline model, then in both cases reliability converges from above to \bar{r}_{crit} as depth increases:

$$\lim_{d \rightarrow \infty} \mathcal{R}_x^d(1) = \lim_{d \rightarrow \infty} \mathcal{R}_x^d(r_A) = \bar{r}_{\text{crit}}.$$

Thus, if $r_A \geq \bar{r}_{\text{crit}}$, the proof of Theorem 1 from the paper and Proposition 3 from the paper go through for the r_A -baseline model. In particular, there is equilibrium fragility for the same parameters.

Now consider the case of $r_A < \bar{r}_{\text{crit}}$. We will split the proof into two cases according to the limit value of x . If we have a sequence of equilibria with limit investment $x \leq x_{\text{crit}}$, we can immediately see that the reliability of high-depth firms cannot converge to any $r > \bar{r}_{\text{crit}}$ so the only possibilities are $r = \bar{r}_{\text{crit}}$

or $r = 0$. Thus, any limit equilibrium with investment $x > 0$ must be fragile. Alternatively, if we have a sequence of equilibria with limit investment $x > x_{\text{crit}}$ then in the limit reliability must be greater than r_{crit} . (By Lemmas SA2 and SA11, the only feasible limit reliability less than r_{crit} is 0 and only the limit investment x_{crit} can yield limit reliability r_{crit} .) Hence

$$\lim_{d \rightarrow \infty} \mathcal{R}_x^d(1) = \lim_{d \rightarrow \infty} \mathcal{R}_x^d(r_A) > r_{\text{crit}}.$$

As such investments are not a limit equilibrium of the baseline model, they cannot be a limit equilibrium of the r_A -baseline model. Putting these cases together, we conclude that if $x^* > 0$ then the equilibrium must be fragile.

We now argue that in a equilibrium of the partial depth knowledge model the equilibrium behavior of the high-depth firms must be the same as equilibrium behavior in the adjusted baseline model. In a symmetric undominated equilibrium of the partial depth knowledge model the low-depth firms make the same investment. Thus all low-type firms at depth \bar{d}_l have the same reliability. Denote this reliability \bar{r} . In such an equilibrium the high-depth firms take this reliability of the low-depth firms as given. Thus the high-depth firm problem is identical to the problem facing firms in the adjusted baseline model, where we set $r_A = \bar{r}$ and let depth d in the adjusted baseline model correspond to depth $\bar{d}_l + d$ for the high-depth firms in the partial depth knowledge model.

We thus conclude, by our analysis of the r_A -adjusted baseline model, that in any symmetric undominated equilibrium of the partial depth knowledge model if there is positive limit investment in equilibrium then there is also equilibrium fragility, as claimed. \square

SA3. MICROFOUNDATIONS (FOR ONLINE PUBLICATION)

In this appendix we use a canonical production network model with monopolistic competition to microfound the functionality of firms, firms' profit functions and the planner's objective function—i.e., Properties A, B, B', and C (all in the paper). Our goal is to keep this part of the model standard, in order to put the focus on the structure of the underlying supply network. The proofs in Appendix SA2 above depend on the structure of production only through the properties just mentioned.

We first study production after investment has occurred and a given realized supply network is in effect; we then discuss the modeling of investment in a prior stage.

SA3.1. Production. In this subsection we provide a model of production that microfound the functionality of firms.

SA3.1.1. Intermediate and final versions of each variety. We let production of any variety $v \in \mathcal{V}_i$ be used in one of two ways. First it can be transformed into an intermediate good version, which is usable only by those varieties v' such that $v \in PS_i(v')$ —i.e., the varieties v' for which v is a potential supplier. This can be interpreted as a costless transformation made possible by the supply relationship with v' that makes v suitable for use by v' . Alternatively, v can be converted costlessly into a different, consumption good version, denoted \underline{v} . (As we will discuss later, this transformation technology is owned by a particular firm, which earns rents from selling this differentiated consumption good.)

SA3.1.2. Quantities and production functions. Suppose v procures for its production $z_{v,v'}$ units of the variety $v' \in S_j(v)$. For a given required input $j \in I(i)$, let $z_{v,j}$ be the total amount of j sourced by v , summing across all of v 's suppliers for this input, and write \mathbf{z}_v for the vector of all these quantities associated with variety v . Let ℓ_v be the amount of labor used by variety v . Labor is the only factor (i.e., unproduced good) and it is inelastically supplied, with $\bar{\ell} = 1$ unit of it. The output of v is

$$\phi_v = f(\ell_v, \mathbf{z}_v) := (\ell_v)^{\varepsilon_\ell} \prod_{j \in I(i)} (z_{v,j})^{\varepsilon_j}, \quad (\text{SA-14})$$

where $\varepsilon_\ell + |I(i)|\varepsilon_z = 1$ so that there are constant returns to scale. Thus, all varieties in any set $S_j(v)$ are perfect substitutes. Concerning substitutability *across* varieties, note that production is not possible if one of the inputs cannot be sourced, but on the intensive margin different inputs are substitutable.¹¹

¹¹We view the assumption that all inputs must be sourced for successful production as mild. Indeed, Boehm, Flaaen, and Pandalai-Nayar (2019) estimate, in the context of a supply network disruption event, a production function with low elasticity of substitution, closer to Leontief than Cobb–Douglas.

Let $q_{\underline{v}}$ be the quantity of \underline{v} consumed. The household consumes aggregate gross production less investments in reliability which are paid for in the aggregated final goods. Aggregate gross production of a *consumption good* is given by

$$Y = \left(\int_{\mathcal{V}} (q_{\underline{v}})^{\eta_C} d\underline{v} \right)^{1/\eta_C}, \quad (\text{SA-15})$$

a demand aggregator with $\frac{1}{2} < \eta_C < 1$.¹² Y_C of the consumption good is allocated to consumption, while Y_I is allocated to investment. Household utility is Y_C . Given investments, efficiency corresponds to maximizing Y .

SA3.1.3. *Functionality.* Equation (SA-14) implies that variety $v \in \mathcal{V}_i$ is able to produce if and only if it can procure quantities $(z_{v,v'})_{v,v' \in \mathcal{V}}$ from its suppliers such that $z_{v,j} > 0$ for all $j \in I(i)$. This will be possible if and only if it has at least one link in the realized supply network to a functional supplier for each input it requires—in other words, if and only if it is functional (as defined in Section IC in the paper). Note that depth-zero nodes have the same production functions as any other nodes; the only difference is that they are not reliant on the operation of specific supply relationships, and always have a functional firm to source their inputs from.

SA3.2. *Equilibrium on a realized supply network.* We now define a competitive equilibrium and study its structure on a realized supply network. Given a realized supply network, functional firms choose how much of each variety of each input to source (taking prices as given) and what price to set for its consumer good, while the representative consumer supplies a unit of labor inelastically and chooses how much of each consumer good variety to buy (taking prices as given). We start by considering the consumer's problem, and then consider firms' problems before using these to define an equilibrium.

SA3.2.1. *Equilibrium: Definitions.* Let w denote the wage; p_v the price of variety v when used as an intermediate; and $p_{\underline{v}}$ the price paid by the consumer for the final good corresponding to variety v . The numeraire is the price paid by the consumer for the final output.

As Y_I , the amount of final good devoted to investment, is sunk at the production stage, the household's problem is equivalent to choosing final good consumptions $(q_{\underline{v}})_{\underline{v}}$ to maximize equation (SA-15) subject to the budget constraint

$$\int_{\mathcal{V}} p_{\underline{v}} q_{\underline{v}} d\underline{v} \leq w + \int_{\mathcal{V}} \Pi_v d\underline{v}, \quad (\text{SA-16})$$

where the right hand side is labor income from the consumer's inelastically supplied unit of labor at wage w and income from the profits Π_v of all firms (including those firms that are not functional). Let $q^*(\underline{p})$ denote the unique input bundle maximizing final good production, where \underline{p} is a vector of the final good prices set by different varieties, and let $q_{\underline{v}}^*(\underline{p})$ denote the corresponding amount of variety v demanded in the production of the final good.

Each firm takes the prices of all intermediate goods as given, as well as the final good prices of other varieties. Prices for intermediate goods are competitive, i.e., equal to marginal costs of production.¹³

Our model is one of monopolistic competition in final goods: a functional firm's problem is to choose input quantities $(z_{v,v'})_{v,v' \in \mathcal{V}}$ from its functional suppliers in the realized supply network, labor demand ℓ_v and a price for its consumer good \underline{v} to maximize its profits.¹⁴ We use efficient pricing for interfirm transactions to focus on inefficiencies coming from network investment, rather than multiple-marginalization issues studied elsewhere, but a model of this form could accommodate wedges in the interfirm prices if desired.

¹²The same results would hold if we defined a suitable nested-CES aggregator, with categories of products of equal measures being the nests. The upper bound on η_C ensures that the elasticity of substitution between varieties is finite (so that variety is valuable), while the lower bound ensures that the equilibrium returns to increasing variety are concave, which is technically convenient; see the discussion after Definition 1 in the paper.

¹³Thus, the intermediate of variety v has a single price irrespective of who buys it. This pricing assumption amounts to requiring efficient production, with no distortions even when a supplier has (in a given realization) market power over a buyer. As documented by (Uzzi, 1997; Kirman and Vriend, 2000), avoiding hold-up is an important function of relational contracts in practice.

¹⁴As in standard monopolistic competition models, a firm commits to a price $p_{\underline{v}}$ for its consumption good \underline{v} before producing. These goods are sold at a markup above marginal cost; the quantity is determined by consumer demand at that price. As there are constant returns to scale, the amount of intermediate goods produced has no bearing on production costs for the final goods, and vice versa.

It is convenient to break this problem into two steps. First, for any quantity a firm produces it must choose its inputs to minimize its cost of production. As there are constant returns to scale, this requires choosing inputs to minimize the cost of producing one unit and then scaling these choices up or down to meet the production target. Thus the relative amounts of different inputs and labor used must solve

$$\min_{z_v, \ell_v} \sum_j \sum_{v' \in S_j(v)} p_{v'} z_{v, v'} + w \ell_v \quad \text{subject to } f(\ell_v, z_v) = 1 \quad (\text{SA-17})$$

Let $z_{v, v'}^*$ denote the quantity of variety v' that v chooses to source per unit of its output, and ℓ_v^* the amount of labor that v sources per unit of its output.

Given the unit cost of production generated by solving (SA-17), each variety v sets a price for its consumer good that maximizes its profits (taking reliability investment costs as sunk) and hence solves

$$\max_{p_v} q_v^*(p)[p_v - \sum_{v'} p_{v'} z_{v, v'}^* - w \ell_v^*] \quad (\text{SA-18})$$

Given a realized supply network $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$, with functional firms \mathcal{V}' , a competitive equilibrium is given by a specification of intermediate good production $z_{v, v'}$ for all $(v, v') \in \mathcal{E}'$, final good production q_v for all $v \in \mathcal{V}'$, wage w , intermediate good price p_v for all $v \in \mathcal{V}'$ and final good price p_v for all $v \in \mathcal{V}'$ such that the following conditions hold:

- the household is choosing how much to consume of each consumer good variety given the prices of these varieties to maximize its utility (SA-15) subject to its budget constraint (SA-16);
- all functional varieties $v \in \mathcal{V}'$ are choosing inputs in ratios that minimize their unit cost given input prices (solve (SA-17));
- all functional varieties $v \in \mathcal{V}'$ are choosing consumer good prices p_v that maximize their profits given other consumer good prices (solve (SA-18));
- markets clear ($\phi_v = \sum_{v' \in \mathcal{V}'} z_{v, v'} + q_v^*$ for all $v \in \mathcal{V}'$).

SA3.2.2. Equilibrium: Analysis. Equilibrium is characterized by all firms producing the same quantities of their respective final goods and selling these at the same price. While they affect the propagation of production failures, conditional on being functional local network features do not alter equilibrium prices and quantities. This can be seen by analyzing the prices of different intermediate goods. First, because of constant returns to scale, firms sell their intermediate goods at a price equal to marginal cost. It turns out that within an industry each functional variety is subject to the same intermediate costs, and hence sells at the same intermediate price regardless of their depth. Second, because of the regularity of the supply network, a symmetry exists across industries which results in them selling at the same intermediate prices. These results are shown formally in Lemmas SA12 and SA13 respectively. In Lemma SA14, we show these results imply that all firms sell the same amounts to consumers at the same final prices.

LEMMA SA12. The selling prices p_v of the intermediate goods of all varieties $v \in \mathcal{V}_i$ within any industry i are equal to some common price p_i .

Proof. As there are constant returns to scale, in equilibrium each intermediate good must be sold at a price equal to their respective marginal costs of production. Thus, by constant returns to scale, the price of each variety's intermediate good, p_v , does not depend on the quantity firm v produces. We establish the conclusion of the lemma by induction on the depth of the variety v . If $d(v) = 0$, firm $v = (i, f)$ can source from any firm in each industry $j \in I(i)$ and can thus minimize its marginal costs. It is easy to deduce from this that $p_v = \min_{f \in [0, 1]} p_{(i, f)}$. Now take a variety v of depth $d(v) = d$; by definition v sources its intermediate goods from firms with depth $d - 1$ in each industry j . If these produce at minimal cost so it follows that $p_v = \min_{f \in [0, 1]} p_{(i, f)}$. This gives us that all firms in the same industry have the same marginal costs and thus the same intermediate prices, regardless of their depth. \square

LEMMA SA13. For any two industries i and j their intermediate prices are equal: $p_i = p_j$.

Proof. From Lemma SA12, there is a price p_i such that all firms in industry i sell intermediate goods at price p_i . By constant returns to scale, p_i is not a function of the quantity produced and is only a function of the prices of the intermediates p_j where $j \in I(i)$. Moreover, each industry requires the same number of inputs and combines these with labor (at common wage w). Thus marginal cost of production for a

firm and hence its price can be expressed as a function of just its intermediate good input prices p_j for $j \in I(i)$ and the wage w and by symmetry this is the same function for all industries. We denote this function by \mathfrak{P} ; it takes as its input m intermediate good prices and the wage. The function is invariant to permutations of intermediate good prices. We can construct a vector of input prices for industry i as $\mathbf{M}_i \mathbf{p}$, where \mathbf{M}_i is a $m \times N$ matrix such that for each row r , the entry $M_{r,k} = 0$ for all k but one which we label $j_i(r)$. This index satisfies $j_i(r) \in I(i)$; $\mathbf{M}_{r,j_i(r)} = 1$; and $\mathbf{M}_{r',j_i(r)} = 0$ for $r' \neq r$. (The matrix simply selects, in each row, one of the inputs of industry i , and the corresponding entry of $\mathbf{M}_i \mathbf{p}$ corresponds to its price.) Then, for all industries i ,

$$p_i = \mathfrak{P}(\mathbf{M}_i \mathbf{p}; w),$$

The function \mathfrak{P} is weakly increasing in each input price and strictly increasing in each intermediate input price p_j for $j \in I(i)$.

Now assume, for the sake of contradiction, that not all intermediate input prices are the same. Without loss of generality, let $p_a = \max\{p_1, \dots, p_N\}$ and $p_b = \min\{p_1, \dots, p_N\}$ with $\frac{p_a}{p_b} = r > 1$. By definition, the price of each intermediate for industry a must be at most p_a , and similarly the price of each input for industry b must be at least p_b . Thus each element of $\mathbf{M}_a \mathbf{p}$ must be less than or equal to each element of $r \mathbf{M}_b \mathbf{p}$. Now, we claim

$$p_a = \mathfrak{P}(\mathbf{M}_a \mathbf{p}; w) \leq \mathfrak{P}(r \mathbf{M}_b \mathbf{p}; w) < r \mathfrak{P}(\mathbf{M}_b \mathbf{p}; w) = r p_b = p_a.$$

To show the strict inequality, note that strictly increasing all intermediate prices relative to the wage leads to a substitution toward labor, making the increase in unit costs less than proportional. This is a contradiction. Thus, all industries must sell intermediate goods at the same price. \square

LEMMA SA14. For any two functional firms, $v, v' \in \mathcal{V}'$, the prices of these firms' final goods are equal and the same amount of each good is sold: $p_v = p_{v'}$ and $q_v^*(p) = q_{v'}^*(p)$.

Proof. Recall that $q_v^*(p)$ denotes the demand of the consumer for final good v at prices p . We first calculate the price elasticity of this demand for final good v .

The consumers maximize (SA-15) subject to the constraints of (SA-16). Using the method of Lagrange multipliers to solve for the quantity of each variety demanded in terms of prices gives that for any two varieties v, v' ,

$$\frac{q_v^*(p)}{p_v^{1/(\eta_C - 1)}} = \frac{q_{v'}^*(p)}{p_{v'}^{1/(\eta_C - 1)}} \quad (\text{SA-19})$$

Because there is a continuum of firms, changing the price level and quantity of firm v will not affect the right-hand side of equation (SA-19). Thus, the left-hand side of equation (SA-19) is a constant function of p_v . Upon differentiating this function with respect to p_v , we get

$$\frac{\partial q_v^*}{\partial p_v} \cdot p_v - \frac{q_v^*}{\eta_C - 1} = 0$$

Rearranging gives us that the price elasticity of demand for variety v at any price level is $\frac{1}{\eta_C - 1}$.

Now consider a firm's pricing problem. Given an intermediate good price p (which by Lemmas SA12 and SA13 is constant for all intermediate goods), there is a unique solution to the cost minimization problem of each firm (Equation (SA-17)).

This marginal cost of production is independent of the quantity produced and is faced by all firms in each industry. Denote this marginal cost c . At a solution to this problem the firm charges a price p_v that satisfies the Lerner condition:

$$\frac{p_v - c}{p_v} = -\frac{1}{\epsilon_d(p_v)}, \quad (\text{SA-20})$$

where ϵ_d is the price elasticity of demand as a function of final good prices. Rearranging equation (SA-20) and substituting the previously calculated elasticity, we have

$$\eta_C p_v = c \quad (\text{SA-21})$$

It is clear that the left-hand side is injective in p_v and there is a unique profit-maximizing price that all firms choose. Additionally, as the price elasticity of demand is constant, equation (SA-20) implies that all firms will charge the same constant markup regardless of the quantity of goods they are producing. \square

We have shown that in equilibrium: (i) each functional variety v has the same consumer good output q_v , which we will call q ; (ii) these goods are all priced at the same price p_v ; (iii) all intermediate goods of variety $v \in \mathcal{V}$ have the same price $p_v = p$. There are two other features of the equilibrium that are crucial:

LEMMA SA15.

- (1) Gross output is equal to $h(\rho(x, \mu))$, where $h : [0, 1] \rightarrow \mathbb{R}_+$ is an increasing and continuous function with bounded derivative and $h(0) = 0$.
- (2) The expected gross profit of producing each variety, conditional on that variety being functional, is equal to $g(\rho(x, \mu))$; where $g : [0, 1] \rightarrow \mathbb{R}_+$ is a decreasing function.

Proof. Note that because labor is the only unproduced input, all labor can be assigned to the production of final goods by allocating labor used to produce intermediate goods to the final good that this intermediate good is ultimately used to produce. Let L_v be the total labor assigned in this way to the production of variety v 's final good. As by Lemmas SA13 and SA14 all firms have equivalent production functions, face the same input prices and choose the same final good prices, they will use the same ratio of different inputs in production. This symmetry throughout the supply network implies that $L_v = L$ for all varieties v . (This does not depend on depth because the zero depth producers have production functions identical to those of other firms and buy inputs—they can just do so from any variety without needing a specific relationship.) As there is a total supply of labor equal to one and it is supplied inelastically, the amount of labor that can be assigned to the production of any given final good is $1/\lambda(\mathcal{V}')$, where $\lambda(\mathcal{V}')$ is the measure of functional firms (and hence equal to $\rho(x, \mu)$ as the measure of firms is 1). As L units of labor are required to produce one unit of any final good, this implies that

$$q_v^* = q^* = \frac{1}{L\lambda(\mathcal{V}')} \quad Y = (\lambda(\mathcal{V}')(q^*)^{\eta_C})^{1/\eta_C} = \frac{\lambda(\mathcal{V}')^{1/\eta_C-1}}{L} = \frac{\rho(x, \mu)^{1/\eta_C-1}}{L} \quad (\text{SA-22})$$

Setting

$$h(\rho(x, \mu)) = \frac{\rho(x, \mu)^{1/\eta_C-1}}{L},$$

completes the proof of part (1).

It can be computed that L depends only on the constants in the production functions, and not on x , which establishes (3).¹⁵

To compute the wage w in terms of the numeraire, we express the aggregate expenditure on final goods in two different ways. On the one hand as the price of the aggregate consumer good has been normalized to 1, it is simply the quantity of aggregate output as given by equation (SA-22) On the other hand, it is the expenditure of the consumer on all firms' final goods, which we simplify using equation (SA-21), and noting that as all inputs are sold at cost all firms have a marginal cost of production equal to wL :

$$p_v \int_{\mathcal{V}} (q_v) dv = \rho(x, \mu) \frac{wL}{\eta_C} \frac{1}{\rho(x, \mu)L} = \frac{w}{\eta_C}$$

Combining these two expressions for expenditure implies that

$$w = \left(\frac{\eta_C}{L}\right) \rho(x, \mu)^{\frac{1-\eta_C}{\eta_C}}.$$

From equation (SA-21) each firm sets a final good price of $p_v = wL/\eta_C$. Thus each firm earns gross profits

$$(p_v - c) q^* = w \left(\frac{1 - \eta_C}{\eta_C}\right) \frac{1}{\lambda(\mathcal{V}')} = \left(\frac{\eta_C}{L}\right) \rho(x, \mu)^{\frac{1-\eta_C}{\eta_C}} \left(\frac{1 - \eta_C}{\eta_C}\right) \frac{1}{\rho(x, \mu)} = \left(\frac{1 - \eta_C}{L}\right) \rho(x, \mu)^{\frac{1}{\eta_C} - 2}. \quad (\text{SA-23})$$

This directly implies the second part of the lemma, using the fact that $\eta_C \in (\frac{1}{2}, 1)$. \square

¹⁵Let ℓ be the amount of labor firm v directly hires to produce a unit of its final good. Let z be the amount of each intermediate firm v uses from one of its suppliers to produce a unit of its final good. By symmetry and constant returns to scale, this firm minimizes its unit cost in terms of labor, $L = \frac{\ell}{1-mz}$ subject to the constraint that $1 = z^{m\varepsilon_z} \ell^{\varepsilon_\ell}$. Solving the Lagrangian yields that for some constant γ , we have $\frac{\varepsilon_z}{z} = \frac{\gamma\ell}{(1-mz)^2}$ and $\frac{\varepsilon_\ell}{\ell} = \frac{\gamma}{1-mz}$. Solving this shows that at the optimum $z = \varepsilon_z$, we have $\ell = \varepsilon_z^{-m\varepsilon_z/\varepsilon_\ell}$ and $L = \frac{\varepsilon_z^{-m\varepsilon_z/\varepsilon_\ell}}{\varepsilon_\ell}$.

Lemma SA15 provides microfoundations for Properties A and C from the paper. Setting $Y(\mathcal{V}') = h(\rho(x, \mu))$, part (1) shows that maximized aggregate output $Y(\mathcal{V}')$ satisfies Property A from the paper. Part (2) shows that the expected gross profit function takes the form $g(r)$ assumed in the model, with $g(r)$ satisfying Property C from the paper—i.e., that $g(r)$ is a decreasing, continuously differentiable function. For these statements it is important that L depends only on constants, not on x , which we have shown.

Part (2) of Lemma SA15 follows from the household's love of variety. Because of the love of variety that households have, the profit maximizing price that firms set for their consumer goods is at a markup over marginal cost. On the other hand, constant returns to scale in the production function means that intermediate goods that can be produced (i.e., which are able to source all of their required inputs) must be priced at marginal cost in equilibrium. So firms' profits depend just on the markup they charge to consumers, and the quantities they sell to consumers. When more other varieties are functional the consumer's love of variety reduces demand for a given variety and hence the profits obtained by that variety.

SA3.3. Endogenizing relationship strengths: Efficient and equilibrium outcomes. Now we turn to the earlier stage, before production, where firms invest in relationship strengths. Here our goal is to show how the reduced-form modeling of investment costs that we presented in the main text (Section IIB) fits into the general equilibrium model.

SA3.3.1. The technology of investment. There is a technology available to each firm to invest the consumption good and increase relationship strengths. Before production, each firm if pledges some amount of the consumption good to invest. We posit that it costs a firm $\frac{1}{\kappa}c(x_{if} - \underline{x})$ units of the numeraire (the consumption good) to achieve relationship strength x with its potential suppliers, where c is a given function—an exogenous technology of investment.

Thus, a firm's expected profit when it is making its investment decision is

$$\Pi_{if} = P(x_{if}; x, \mu)g(r) - \frac{1}{\kappa}c(x_{if} - \underline{x});$$

here both expected revenues and costs are in units of the numeraire. This microfounds the form of the firm profits we have assumed.

SA3.3.2. Planner's problem. We study the problem of efficiently choosing the investments in the first stage. The planner may choose any symmetric investments x for the firms. For a fixed choice of relationship strengths x , let $h(\rho(x, \mu))$ denote the gross production of the aggregated consumption good. As we have said, household consumption is $Y_C = Y - Y_I$, where Y_I is the value of the investment in relationship strength in terms of the quantity of the consumption good devoted to it. These are costs that are sunk prior to production. Obtaining reliability x for all such relationships costs the planner

$$Y_I = \sum_{i \in \mathcal{I}} \int_{v \in \mathcal{V}_i} \frac{1}{\kappa}c(x - \underline{x})dv = \frac{1}{\kappa}c(x - \underline{x}).$$

(Recall that the measure of the set of all varieties is $\lambda(\mathcal{V}) = 1$.) Setting $c_P(x) = c(x - \underline{x})$ the planner's cost function inherits the key properties of the individual cost function (it is continuous, increasing, and weakly convex with $c(0) = 0$, $c'(0) = 0$ and $\lim_{x \rightarrow 1} c(x) = \infty$ by Property B' in the paper) and hence satisfies the properties assumed in Property B in the paper.

Maximizing $Y_C = Y - Y_I$ amounts to our planner's problem from the main text:

$$\max_{x \in [0, 1]} h(\rho(x, \mu)) - \frac{1}{\kappa}c_P(x)$$

SA4. TECHNICAL RESULTS

In this section we restate the results for which proofs were omitted, and then provide the missing proofs.

SA4.1. **Lemma SA1.** Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ recall the definition

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (\text{SA-24})$$

Then there are values $x_{\text{crit}}, \bar{r}_{\text{crit}} \in (0, 1]$ such that:

- (i) $\widehat{\rho}(x) = 0$ for all $x < x_{\text{crit}}$;
- (ii) $\widehat{\rho}$ has a (unique) point of discontinuity at x_{crit} ;
- (iii) $\widehat{\rho}$ is strictly increasing for $x \geq x_{\text{crit}}$;
- (iv) the inverse of $\widehat{\rho}$ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[\bar{r}_{\text{crit}}, 1]$, where $\bar{r}_{\text{crit}} = \widehat{\rho}(x_{\text{crit}})$;
- (v) χ is positive and quasiconvex on the domain $(0, 1]$;
- (vi) $\chi'(\bar{r}_{\text{crit}}) = 0$.

Proof. We first list some properties of $\widehat{\rho}$ and χ .

Property 0: For positive r in the range of $\widehat{\rho}$, we have $r = \widehat{\rho}(x)$ if and only if

$$x = \frac{1 - (1 - r^{1/m})^{1/n}}{r}.$$

This is shown by rearranging equation (SA-1).

Property 1: $\chi(1) = 1$. This follows by inspection.

Property 2: $\chi(r) > 0$ for all $r \in (0, 1]$. This follows by inspection.

Property 3: $\lim_{r \downarrow 0} \chi(r) = \infty$. This follows by an application of l'Hopital's rule, i.e.

$$\lim_{r \downarrow 0} \frac{\frac{d}{dr} \left(1 - (1 - r^{1/m})^{1/n}\right)}{\frac{d}{dr} r} = \lim_{r \downarrow 0} \frac{(1 - r^{1/m})^{1/n-1} r^{1/m-1}}{mn} = \infty.$$

Property 4: $\lim_{r \uparrow 1} \chi'(r) = \infty$. This follows by examining

$$\chi'(r) = \frac{r^{1/m} (1 - r^{1/m})^{1/n-1}}{mn r^2} + \frac{(1 - r^{1/m})^{1/n} - 1}{r^2}$$

Property 5: There is a unique interior $\bar{r}_{\text{crit}} \in (0, 1)$ minimizing $\chi(r)$. To show this, define $z(r) = 1/(1 - r^{1/m})$, and note that $r \in (0, 1)$ satisfies $\chi'(r) = 0$ if and only if the corresponding $z(r) > 1$ solves

$$z - 1 = mn(z^{1/n} - 1);$$

here we use that the function z is a bijection from $(0, 1)$ to $(1, \infty)$. The equation clearly has exactly one solution for $z > 1$.¹⁶ Now it remains to see that the unique $r \in (0, 1)$ solving $\chi'(r) = 0$ defines a local minimum. Note that χ is continuously differentiable on $(0, 1)$. Property 3 implies that $\chi'(r) < 0$ for some $r < \bar{r}_{\text{crit}}$ while Property 4 implies that $\chi'(r) > 0$ for some $r > \bar{r}_{\text{crit}}$. These points suffice to show Property 5.

To prove the lemma, we relate desired properties of $\widehat{\rho}$ to properties of χ ; the claims made here can be visualized by referring to Figure 2, panels (A) and (B), which illustrate the properties of the functions involved.

Together, Properties 0 and 5 imply that there is no $r > 0$ in the range of $\widehat{\rho}$ such that $x < \chi(\bar{r}_{\text{crit}})$. Let $x_{\text{crit}} = \chi(\bar{r}_{\text{crit}})$, which yields (vi) of the lemma; then what we have said implies $\widehat{\rho}(x) = 0$ for $x < x_{\text{crit}}$, i.e., statement (i) of the lemma. The proof of Property 5 also implies statement (v) in the Lemma.

It remains to show (ii-iv) of the Lemma. By definition, $\widehat{\rho}(x)$ is the largest solution of (SA-1). Properties 1 and 5 imply that on the domain $[\bar{r}_{\text{crit}}, 1]$, χ is a strictly increasing function whose range is $[x_{\text{crit}}, 1]$. Fix an $r \in [\bar{r}_{\text{crit}}, 1]$ and let $x = \chi(r)$. What we have said implies that (x, r) solves (SA-1) and that there is no r' with $r' > r$ such that (x, r') solves (SA-1). Thus, by definition $r = \widehat{\rho}(x)$. Notice that as we vary r in the interval, x varies over the interval $[x_{\text{crit}}, 1]$. This establishes (iv), and (iii) follows immediately from

¹⁶The left-hand side is linear and the right-hand side is concave, since $n \geq 2$. At $z = 1$ the two sides are equal, and the curves defined by the left-hand and right-hand sides are not tangent, so there is exactly one solution $z > 1$.

the fact that χ is increasing on the domain in question. For (ii), it suffices to deduce from Properties 2 and 5 that the minimum of χ has both coordinates positive.

We now consider the case $n = 1$. Here, $\frac{d\chi(r)}{dr} = \frac{r^{1/m} - mr^{1/m}}{mr^2} < 0$ for all $r \in (0, 1]$. Since from Properties 1 and 3 (which still hold when $n = 1$), $\chi(1) = 1$ and $\lim_{r \downarrow 0} \chi(r) = \infty$, it follows that $\chi(r)$ is decreasing in r and has image $[1, \infty)$. Thus, by logic similar to the above, $\widehat{\rho}(x) = 0$ for all $x \in [0, 1)$ and $\widehat{\rho}(x) = 1$ when $x = 1$. It follows that $x_{\text{crit}} = 1$ in this case and all the statements of the lemma are satisfied, though some of them are trivial. \square

SA4.2. Lemma SA3. Fix any $m \geq 2$, $n \geq 2$, and $r \geq r_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n , and x) such $0 \leq x_1 < x_2 < 1/r$ and so that:

0. $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$;
1. $Q(x_{if}; r)$ is increasing and convex in x_{if} on the interval $[0, x_1]$;
2. $Q(x_{if}; r)$ is increasing and concave in x_{if} on the interval $(x_1, x_2]$;
3. $Q(x_{if}; r)$ is decreasing in x_{if} on the interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

Proof. As a piece of notation, define

$$\zeta(x_{if}; r) = 1 - x_{if}r.$$

When using ζ , we will often omit the arguments for brevity. Then

$$Q(x_{if}; r) = mn r \zeta^{n-1} (1 - \zeta^n)^{M-1}.$$

Statement 0. It follows immediately from this equation that $Q(0; r) = Q(1/r; r) = 0$ and $Q(x_{if}; r) > 0$ for all $x_{if} \in (0, 1/r)$. This establishes Claim 0 in the lemma statement.

Statements 1–3. Establishing Statements 1–3 is more involved; we begin by studying the first derivative of Q to establish the increasing/decreasing statements, and then move to the second derivative to establish the convex/concave statements.

We can calculate

$$Q'(x_{if}; r) = -mn r^2 \zeta^{n-2} (1 - \zeta^n)^{M-2} [(mn - 1) \zeta^n - n + 1].$$

Note that for $x_{if} \in (0, 1/r)$ we have¹⁷

$$\text{sign}[Q'(x_{if}; r)] = \text{sign}[(mn - 1) \zeta^n - n + 1]. \quad (\text{SA-25})$$

Further, for sufficiently small $x_{if} > 0$

$$\text{sign}[(mn - 1) \zeta^n - n + 1] = \text{sign}[n(m - 1)] > 0.$$

Thus $Q'(0; r) > 0$.

We will now deduce from the above calculations about Q' that there is exactly one local maximum of $x_{if} \mapsto Q(x_{if}; r)$ on its domain, $[0, 1/r]$. First, as this is a continuous function with $Q(0; r) = 0 = Q(1/r; r)$ and $Q'(0; r) > 0$, it follows there is an interior maximum of $Q(x_{if}; r)$ in the interval $(0, 1/r)$. Next, by (SA-25), $\text{sign}[Q'(x_{if}; r)] > 0$ if and only if

$$x_{if} < \frac{1 - \left(\frac{n-1}{mn-1}\right)^{\frac{1}{n}}}{r}.$$

Thus, there can be at most one value of x_{if}^* with $Q'(x_{if}^*; r) = 0$. Together, these observations imply that Q has one local optimum on its extended domain, which is in fact a global maximum. We let x_2 be defined by the unique value of x_{if} at which $Q'(x_{if}^*; r) = 0$. This establishes Property 3. It also establishes the “increasing” part of Properties 1 and 2, since $Q(x_{if}; r)$ is increasing to the left of x_2 by what we have said.

The next part of the proof studies the second derivative of Q to establish the claims about the convexity/concavity of Q . First note that

$$Q''(x_{if}; r) = mn r^3 \zeta^{n-3} (1 - \zeta^n)^{M-3} H$$

¹⁷The sign operator is +1 for positive numbers, -1 for negative numbers, and 0 when the argument is 0.

where

$$H = \underbrace{(m^2 n^2 - 3mn + 2)}_A \zeta^{2n} + \underbrace{((1 - 3m)n^2 + (3m + 3)n - 4)}_B \zeta^n + \underbrace{n^2 - 3n + 2}_C.$$

For $x_{if} \in (0, 1/r)$, we can see that

$$\text{sign}[Q''(x_{if}; r)] = \text{sign}[H]. \quad (\text{SA-26})$$

Let $z := \zeta^n$ for $z \in (0, 1)$ (which corresponds to $x_{if} \in (0, 1/r)$). We can then write $H = \tilde{H}(z)$ for $z \in (0, 1)$, where

$$\tilde{H}(z) = Az^2 + Bz + C, \quad (\text{SA-27})$$

and A , B and C are constants (labeled above) depending only on m, n . \tilde{H} is therefore a quadratic polynomial in z and its roots depend only on n and m . Further, $A > 0$, $B < 0$, $C > 0$ and $A + B + C > 0$. Thus \tilde{H} is convex in z with $\tilde{H}(0) > 0$ and $\tilde{H}(1) > 0$.

We first argue that $\min_{z \in [0, 1]} \tilde{H}(z) < 0$. Towards a contradiction suppose $\min_{z \in [0, 1]} \tilde{H}(z) \geq 0$. This implies that Q' is nonnegative by equation SA-26 and hence that Q is globally convex. However, we have already established that $Q'(0; r) > 0$, so the convexity of Q implies there can be no interior maximum, which contradicts our deductions above.

An immediate implication of $\min_{z \in [0, 1]} \tilde{H}(z) < 0$ is that $\tilde{H}(z)$ has two real roots, z_1 and $z_2 < z_1$. This establishes the basic shape of $\tilde{H}(z)$ as illustrated in Figure 7.

It will be helpful to sometimes consider the values of x_{if} that correspond to the roots of the $\tilde{H}(z)$. To this end, we define the function

$$X(z) := \frac{1 - z^{1/n}}{r}. \quad (\text{SA-28})$$

We can then set $x_1 = X(z_1)$ (i.e., the first inflection point of Q). Along with what we already know, the deduced shape of $\tilde{H}(z)$ pins down the remaining properties we require about the shape of Q , as we now argue. For an illustration, see Figure 7.

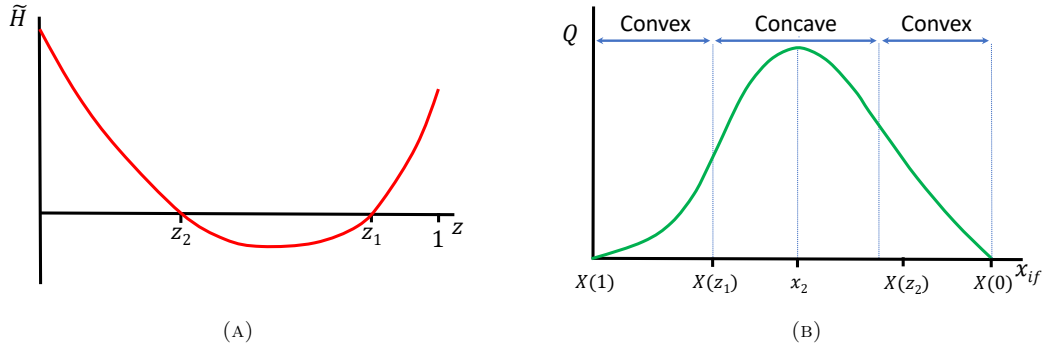


FIGURE 7. Panel (a) shows basic shape of the function $\tilde{H}(z)$. Panel(b) shows the basic shape of the function $Q(x_{if}; r)$, where the convexity and concavity in different regions is implied by the shape of $\tilde{H}(z)$.

As $z_1 > z_2$, it follows that $\tilde{H}'(z_1) > 0$. This corresponds to $Q''(x_{if}; r)$ going from positive to negative as x_{if} crosses $x_1 := X(z_1)$, and thus $Q(x_{if}; r)$ going from convex to concave. As $Q'(0; r) > 0$ and $Q(x_{if}; r)$ is convex for $x_{if} \in [0, x_1]$, the maximum of Q must occur at a value of $x_{if} \in (x_1, 1)$. Recalling that we let x_2 be the value of x_{if} at which $Q(x_{if}; r)$ is maximized we conclude that $x_1 < x_2$. Further, as for values of $x_{if} < x_1$ the function $Q(x_{if}; r)$ is convex, we have established the convexity part of Property 2 of Lemma SA3.

By similar reasoning, $\tilde{H}'(z_2) < 0$. This corresponds to $Q''(x_{if}; r)$ going from negative to positive as x_{if} crosses $X(z_2)$. Recall that x_2 is defined as the maximum of Q . We must have $X(z_2) > x_2$. If not, Q would be increasing and convex for all $x_{if} \geq X(z_2)$, contradicting the existence of an interior maximum as already established. This establishes that the function Q remains concave until after its maximum

point x_2 establishing the concavity part of Property 3. This completes our demonstration of Statements 1–3.

Statement 4. We now argue that $x_1 < x_{\text{crit}}$ to establish Statement 4. We do so in two steps. First we show it for the case in which firms other than *if* have reliability $r = \bar{r}_{\text{crit}}$, and deduce the $r > \bar{r}_{\text{crit}}$ case.

The case $r = \bar{r}_{\text{crit}}$. We need to establish that

$$x_1 = \frac{1 - z_1^{1/n}}{\bar{r}_{\text{crit}}} < \frac{1 - (1 - \bar{r}_{\text{crit}}^{1/m})^{1/n}}{\bar{r}_{\text{crit}}} = x_{\text{crit}}.$$

This holds if and only if $z > 1 - \bar{r}_{\text{crit}}^{1/m}$, which is equivalent to $\bar{r}_{\text{crit}} > (1 - z)^m$. A sufficient condition for this to hold is that $\chi'((1 - z)^m) < 0$, since we know from Lemma SA1 that $\chi'(\bar{r}_{\text{crit}}) = 0$.

Note that

$$\chi'(r) = \frac{\frac{r^{1/m}(1-r^{1/m})^{1/n-1}}{mn} + (1-r^{1/m})^{1/n} - 1}{r^2}$$

Setting $r = (1 - z)^m$ in the above yields

$$\chi'((1 - z)^m) = \frac{\frac{(1-z)z^{1/n-1}}{mn} + (z)^{1/n} - 1}{(1 - z)^{2m}}.$$

The denominator is always positive, hence we only need to verify that the numerator is negative, when evaluated at the larger of the two roots of \tilde{H} , which we will call z_1 . The numerator simplifies to

$$\text{Num}(z) = z^{1/n} \left(\frac{1 - z}{mnz} + 1 \right) - 1$$

and must now be evaluated at the root z_1 . This root is given by the quadratic formula as

$$z_1 = \frac{D + \sqrt{D^2 - 4(n^2 - 3n + 2)(m^2n^2 - 3mn + 2)}}{2(m^2n^2 - 3mn + 2)},$$

where $D = 3mn^2 - 3mn - n^2 - 3n + 4$. This expression for z_1 simplifies to

$$z_1 = \frac{n \left(3m(n - 1) + \sqrt{(m - 1)(n - 1)(5mn - m - n - 7)} - n - 3 \right) + 4}{2(mn - 2)(mn - 1)}.$$

Now we claim that the shape of $\text{Num}(z)$ is pictured in Fig. 8:

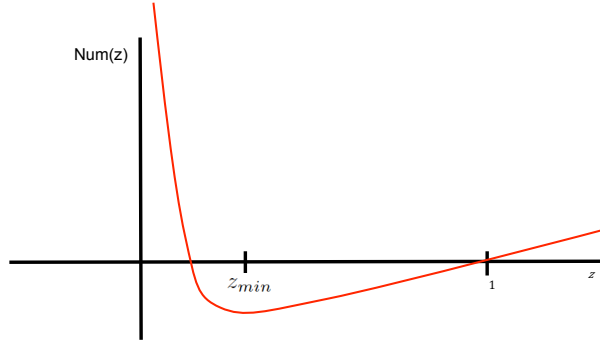


FIGURE 8. Shape of $\text{Num}(z)$.

Thus a sufficient condition for $\text{Num}(z_1) < 0$ is that $z_1 > z_{\text{min}}$.

We will now calculate z_{min} and verify the shape of $\text{Num}(z)$ is as claimed. It is easy to show that $z_{\text{min}} = \frac{n-1}{mn-1}$ by solving $\frac{d\text{Num}(z)}{dz} = 0$.¹⁸ We also note that since $z^{1/n-2}$ and mn^2 are always positive,

$$\text{sign} \left[\frac{d\text{Num}(z)}{dz} \right] = \text{sign} [z(mn - 1) + 1 - n],$$

and thus the function $\text{Num}(z)$ is decreasing on $z \in [0, z_{\text{min}})$ and increasing on $z \in (z_{\text{min}}, \infty)$.

¹⁸Indeed, $\frac{d\text{Num}(z)}{dz} = \frac{z^{1/n-2}(z(mn-1)+1-n)}{mn^2}$ and thus a first-order condition is satisfied when $z(mn - 1) + 1 - n = 0$.

Since $\text{Num}(1) = 0$, the above imply that $\text{Num}(z) < 0$ over the range $[z_{\min}, 1)$. Thus, we only need to check that $z_1 > z_{\min}$ in order to guarantee that $\text{Num}(z_1) < 0$.

It is easy to verify that the following inequality holds¹⁹:

$$\begin{aligned} z_{\min} &= \frac{n-1}{mn-1} \\ &< \frac{n\left(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\right) + 4}{2(mn-2)(mn-1)} = z_1. \end{aligned}$$

We conclude (recalling the beginning of this proof) that $z_1 > 1 - \bar{r}_{\text{crit}}^{1/m}$. This establishes that $\hat{x} < x_{\text{crit}}$ when $r = \bar{r}_{\text{crit}}$.

The case $r > \bar{r}_{\text{crit}}$. From Eq. (SA-28), it is clear that x_1 is decreasing in r . This establishes Property 4 of Lemma SA3, completing the proof of Lemma SA3. \square

SA4.3. Lemma SA7. The function $\beta(r)$ is quasiconcave and has a maximum at $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Proof. Recall that $\beta(r) = mnr^{2-\frac{1}{m}}(1-r^{\frac{1}{m}})^{1-\frac{1}{n}}$. We will show that $\beta(r)$ is quasiconcave by demonstrating that there exists an $\hat{r} \in (0, 1)$ such that $\beta'(r) > 0$ for $r \in (0, \hat{r})$, $\beta'(r) < 0$ for $r \in (\hat{r}, 1)$ and $\beta'(r) = 0$ for $r = \hat{r}$.

$$\begin{aligned} \beta'(r) &= mn \left(2 - \frac{1}{m}\right) r^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}} - \left(1 - \frac{1}{n}\right) mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left(\frac{1}{m} r^{\frac{1}{m}-1}\right) \\ &= mnr^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}} \left[\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) \frac{1}{m} \right] \right]. \end{aligned}$$

Note that the the first factor, $mnr^{1-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{-\frac{1}{n}}$, exceeds 0 for any $r \in (0, 1)$ and is equal to 0 for $r = 0, 1$. Moreover, the expression multiplying it,

$$\left(2 - \frac{1}{m}\right) - r^{\frac{1}{m}} \left[\left(2 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) \left(\frac{1}{m}\right) \right],$$

is a strictly decreasing, continuous function of r . It is also strictly positive when $r = 0$ and strictly negative when $r = 1$, implying that it is equal to 0 at some $r \in (0, 1)$. It follows that there exists an $\hat{r} \in (0, 1)$ satisfying the claimed properties.

Having established that $\beta(r)$ is quasiconcave and $\beta'(\hat{r}) = 0$, it follows immediately that $\beta(r)$ is maximized at $r = \hat{r}$. To find \hat{r} , we use its defining property and solve the following equation:

$$r \left((2m-1)nr^{-1/m} - 2mn + 1 \right) \left(1 - r^{1/m} \right)^{-1/n} = 0.$$

As $r \left(1 - r^{1/m} \right)^{-1/n} > 0$ for any positive production equilibrium, the equation is solved by

$$\hat{r} = \left(\frac{(2m-1)n}{2mn-1} \right)^m.$$

This completes the proof. \square

¹⁹Indeed, multiplying both sides by $2(mn-2)(mn-1)$ yields

$$2(mn-2)(n-1) < n\left(3m(n-1) + \sqrt{(m-1)(n-1)(5mn-m-n-7)} - n - 3\right) + 4$$

which, after some algebra, reduces to

$$0 < n(mn+1-m-n) + n\sqrt{(m-1)(n-1)(5mn-m-n-7)}.$$

Since all terms on the right-hand side are clearly positive when $m, n \geq 2$, the inequality holds.

SA4.4. **Lemma SA8.** Recall that $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$. For all $n \geq 2$ and $m \geq 3$, $\hat{r} < \bar{r}_{\text{crit}}$.

Proof. By Lemma SA1, the function $\chi(r)$ is positive and quasiconvex on the domain $[0, 1]$ with $\bar{r}_{\text{crit}} = \operatorname{argmin}_r \chi(r)$. Thus, $\hat{r} < \bar{r}_{\text{crit}}$ if and only if $\chi'(\hat{r}) < 0$.

Recall that $\chi(r) = \frac{1-(1-r^{1/m})^{1/n}}{r}$ and so

$$\chi'(r) = \frac{\frac{1}{n}(1-r^{1/m})^{1/n-1} \frac{1}{m} r^{1/m} - (1-(1-r^{1/m})^{1/n})}{r^2}.$$

We will study this expression evaluated at $\hat{r} = \left(\frac{(2m-1)n}{2mn-1}\right)^m$ from Lemma SA7.

The denominator of the above expression for $\chi'(r)$ is always positive, so we need only check that the numerator is negative. Calling $A = \frac{n-1}{2mn-1}$ and $B = \frac{n+1-1/m}{n-1}$, we may rewrite the numerator, after some simplifications, as $A^{1/n}B - 1$, and thus we need only check that

$$A^{1/n}B < 1 \tag{SA-29}$$

Let $h(n, m) = A^{1/n}B$. To demonstrate (SA-29), we will show that:

- Step 1: $h(n, 3) < 1$, for all $n \geq 2$.
- Step 2: $h(n, m)$ is decreasing in m , for all $n \geq 2$.

Step 1: First note that

$$h(n, 3) = \left(\frac{n-1}{6n-1}\right)^{1/n} \left(\frac{n+1-1/3}{n-1}\right)$$

We will show that

- Step 1a: $h(n, 3)$ is increasing in n .
- Step 1b: $\lim_{n \rightarrow \infty} h(n, 3) = 1$.

From this we can conclude that $h(n, 3) < 1$, for all $n \geq 2$.

To show Step 1a, note that

$$\frac{\partial h(n, 3)}{\partial n} = -\frac{\left(\frac{n-1}{6n-1}\right)^{1/n} \left((18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n \right)}{3n^2 \frac{(n-1)}{(6n-1)}}$$

The denominator is positive for all $n \geq 2$. Looking at the numerator, since $\left(\frac{n-1}{6n-1}\right)^{1/n} > 0$ for all $n \geq 2$, it suffices to show that

$$(18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n < 0 \quad \text{for all } n \geq 2$$

in order to ensure that $\frac{\partial h(n, 3)}{\partial n} > 0$. It is easy to show that

$$\ln\left(\frac{n-1}{6n-1}\right) < \ln\left(\frac{1}{6}\right) < -1.79$$

Thus, for all $n \geq 2$

$$\begin{aligned} (18n^2 + 9n - 2) \ln\left(\frac{n-1}{6n-1}\right) + 30n^2 + 10n &< -(18n^2 + 9n - 2)1.79 + 30n^2 + 10n \\ &< -32n^2 - 16n + 4 + 30n^2 + 10n \\ &= -2n^2 - 6n + 4 \\ &< 0. \end{aligned}$$

We thus conclude²⁰ that $h(n, 3)$ is increasing in n and Step 1a is proved.

Step 1b follows immediately by noting that

$$\lim_{n \rightarrow \infty} h(n, 3) = \lim_{n \rightarrow \infty} \left(\frac{1}{6}\right)^{1/n} = 1$$

²⁰It is worth noting that this argument does not work for $m = 2$, in which case the numerator of $\frac{\partial h(n, 2)}{\partial n}$ could be negative and thus $h(n, 2)$ could be decreasing.

We have thus proved Step 1.

Step 2. To show that $h(n, m)$ is decreasing in m , let us note that, for all $n \geq 2$

$$\begin{aligned} \frac{\partial h(n, m)}{\partial m} &= B \frac{\partial A^{1/n}}{\partial m} + A^{1/n} \frac{\partial B}{\partial m} \\ &= \left(\frac{n-1}{2mn-1} \right)^{1/n} \left(\frac{-2}{m(n-1)} \frac{mn(1+1/n)-1}{2mn-1} + \frac{1}{(n-1)m^2} \right) \\ &< \left(\frac{n-1}{2mn-1} \right)^{1/n} \left(\frac{-1}{m(n-1)} + \frac{1}{(n-1)m^2} \right) \\ &< 0. \end{aligned}$$

where the second equality follows after some simplifications, while the first inequality follows from the fact that $\frac{mn(1+1/n)-1}{2mn-1} > \frac{1}{2}$, which is easy to check.

We have thus shown that $h(n, m)$ is decreasing in m , for any $n \geq 2$, and Step 2 is thus proved.

This concludes the proof of the lemma. \square

SA5. HOW PRODUCTION UNRAVELS WHEN RELATIONSHIP STRENGTH IS TOO LOW

Figure 3(b) in the main paper shows that when x drops below x_{crit} , the mass of firms that can consistently function falls discontinuously to $\rho(x) = 0$. While we will typically just work with the fixed point as the outcome of interest, the transition will not be instantaneous in practice. How then might the consequences of a shock to x actually play out?

In Figure 9, we work through a toy illustration to shed some light on the dynamics of collapse. Using the same parameters as our previous example, suppose relationship strength starts out at $x = 0.8$. The higher curve in panel (a) is $\mathcal{R}(\cdot; x)$ for this value of x . The reliability of the economy here is r_0 , a fixed point of \mathcal{R} , which is the mass of functioning firms. Now suppose that a shock occurs, and all relationships become weaker, operating with the lower probability $x = 0.7$. The \mathcal{R} curve now shifts, becoming the lower curve.

To consider the dynamics of how production responds, we must specify a few more details. We sketch one dynamic, and only for the purposes of this subsection. We interpret idiosyncratic link operation realizations as whether a given relationship works in a given period. Before the shift in x , a fraction r_0 of the firms are functional. Let $\tilde{\mathcal{F}}(0)$ be the random set of functional firms at the time of the shock to x . Now x shifts to 0.7; we can view this as a certain fraction of formerly functional links failing, at random. Then firms begin reacting over a sequence of stages. Let us suppose that at stage s a firm can source its inputs if it has a functional link to a supplier who was functional in the last stage, $s-1$.²¹ Let $\tilde{\mathcal{F}}(s)$ be the set of these functional firms. By the same reasoning as in the previous subsection, we can see that the mass of $\tilde{\mathcal{F}}(s)$, which we call r_s , is $\mathcal{R}(r_{s-1})$. Iterating the process leads to more and more firms being unable to produce as their suppliers fail to deliver essential inputs. After stage 1, the first set of firms that lost access to an essential input run out of stock and are no longer functional. This creates a new set of firms that cannot access an essential input, and these firms will be unable to produce at the end of the subsequent stage, and so on. The mass of each r_s can be described via the graphical procedure of Figure 9: take steps between the \mathcal{R} curve and the 45-degree line.

This discussion helps make three related points. First, even though the disappearance of the positive fixed point—and thus the possibility of a positive mass of consistently functional firms—is sudden, the implications can play out slowly under natural dynamics.²² The first few steps may look like a few firms being unable to produce, rather than a sudden and total collapse of output.

The second point is more subtle. Suppose that when the dynamic of the previous paragraph reaches r_2 , the shock is reversed, x again becomes 0.8, and \mathcal{R} again becomes the higher curve. Then, with some supply links reactivated, some of the firms that were made non-functional as the supply chain

²¹For example, firms might hold inventories that enable them to maintain production for a certain amount of time, even when unable to source an essential input. Even if firms engage in just-in-time production and do not maintain inventories of essential inputs, there can be a lag between shipments being sent and arriving.

²²Indeed, in a more realistic dynamic, link realizations might be revised asynchronously, in continuous time, and firms would stop operating at a random time when they can no longer go without the supplier (e.g., when inventory runs out). Then the dynamics would play out “smoothly,” characterized by differential equations rather than discrete iteration.

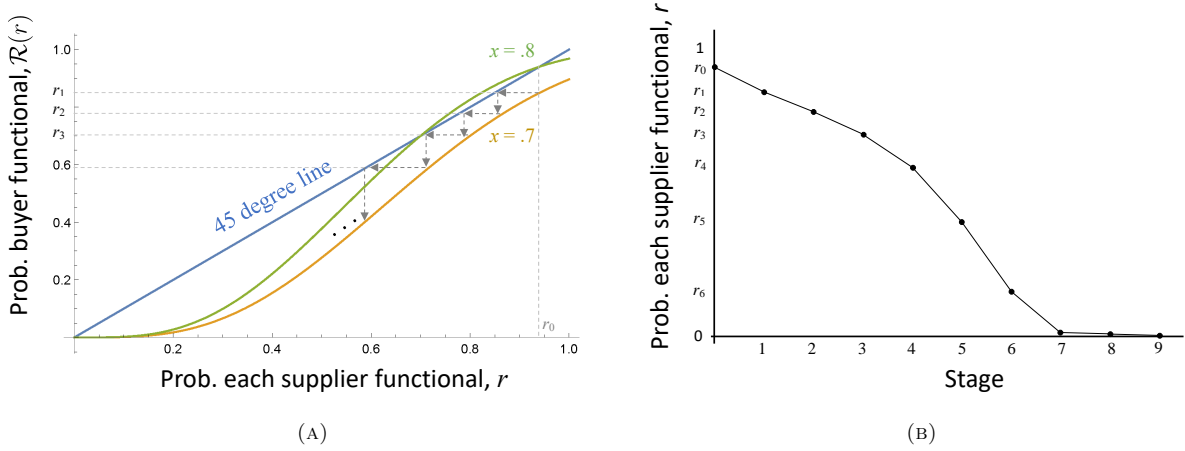


FIGURE 9. The dynamics of unraveling (with the same parameters as in Figure 4 in the main paper, as discussed in Section SA5).

unravelling will become functional again, and this will allow more firms to become functional, and so on. Such dynamics could take the system back to the r_0 fixed point if sufficiently many firms remain functional at the time the shock is reversed. Thus, our theory predicts that *sufficiently persistent* shocks to relationship strength lead to eventual collapses of production, but, depending on the dynamics, the system may also be able to recover from sufficiently transient shocks.

The third point builds on the second. Suppose that a shock is anticipated and expected to be temporary. Then firms may take actions that slow the unravelling to reduce their amount of downtime. For example, they may build up stockpiles of essential inputs. If all firms behave in this way, the dynamics can be substantially slowed down and the possibility of recovery will improve.

Having illustrated some of the basic forces and timing involved in unraveling, we do not pursue here a more complete study of the dynamics of transient shocks, endogenous responses, etc.—an interesting subject in its own right. Instead, from now on we will focus on the size, $\rho(x)$, of the consistent functional set, which is the steady-state outcome under a relationship strength x .

SA6. INTERDEPENDENT SUPPLY NETWORKS AND CASCADING FAILURES

We now posit an interdependence among supply networks wherein each firm's profit depends on the *aggregate* level of output in the economy, in addition to the functionality of the suppliers with whom it has supply relationships. Formally, suppose, $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$, where $K_{\mathfrak{s}}$ is a strictly increasing function and Y is the integral across all sectors of equilibrium output:

$$Y = \int_{\mathcal{S}} \rho(x_{\mathfrak{s}}^*) d\Phi(\mathfrak{s}).$$

Here we denote by $x_{\mathfrak{s}}^*$ the unique positive equilibrium in sector \mathfrak{s} . The output in the sector is the reliability in that sector, $\rho(x_{\mathfrak{s}}^*)$.

The interpretation of this is as follows: When a firm depends on a different sector, a specific supply relationship is not required, so the idiosyncratic failure of a given producer in the different sector does not matter—a substitute product can be readily purchased via the market. Indeed, it is precisely when substitute products are not readily available that the supply relationships we model are important. However, if some sectors experience a sudden drop in output, then other sectors suffer. They will not be able to purchase inputs, via the market, from these sectors in the same quantities or at the same prices. For example, if financial markets collapse, then the productivity of many real businesses that rely on these markets for credit are likely to see their effective productivity fall. In these situations, dependencies will result in changes to other sectors' profits even if purchases are made via the market. Our specification above takes interdependencies to be highly symmetric, so that only aggregate output

matters, but in general these interdependencies would correspond to the structure of an intersectoral input-output matrix, and K would be a function of sector level outputs, indexed by the identity of the sourcing sector.

This natural interdependence can have very stark consequences. Consider an economy characterized by a distribution Ψ in which the subset of sectors with $m \geq 2$ has positive measure, and some of these have positive equilibria. Suppose that there is a small shock to \underline{x} . As already argued, this will directly cause a positive measure of sectors to fail. The failure of the fragile sectors will cause a reduction in aggregate output. Thus $\kappa_s = K_s(Y)$ will decrease in other sectors *discontinuously*. This will take some other sectors out of the robust regime. Note that this occurs due to the other supply chains failing and not due to the shock itself. As these sectors are no longer robust, they topple too following an infinitesimal shock to \underline{x} . Continuing this logic, there will be a domino effect that propagates the initial shock. This domino effect could die out quickly, but need not. A full study of such domino effects is well beyond our scope, but the forces in the very simple sketch we have presented would carry over to more realistic heterogeneous interdependencies.

Fig. 10 shows²³ how an economy with 100 interdependent sectors responds to small shocks to \underline{x} . In this example, sectors differ only in their initial κ 's. The technological complexity is set to $m = 5$ and the number of potential suppliers for each firm is set to $n = 3$. The cost function²⁴ for any firm if is $c(x_{if} - \underline{x}) = \frac{0.01}{(1 - (x_{if} - \underline{x}))^2}$ while the gross profit function is $g(\rho(x)) = 5(1 - \rho(x))$. This setup yields values $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$ delimiting the region corresponding to critical (and therefore fragile) equilibria, as per Theorem 1.

In Fig. 10(a), the productivity shifter of a given sector is distributed uniformly, i.e. $\kappa_s \sim U(\underline{\kappa}, 25)$, so that many sectors have high enough productivity to be in a robust equilibrium while a small fraction have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors thus causes the failure of the fragile sectors (there are initially 12 of them). This then decreases the output Y across the whole economy, but only to a small extent (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is thus only enough to bring one robust sector into the fragile regime and thus to cause it to fail as well upon a small shock. In the end, a total of only 13 sectors have failed.

In contrast, Fig. 10(b) shows an economy where $\kappa_s \sim U(\underline{\kappa}, 13)$, so that more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors causes the failure of the fragile sectors (now initially 24). These have a larger effect on decreasing the output Y across the whole economy (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is now enough to bring many of them into the fragile regime and to cause them to fail upon an infinitesimal shock to \underline{x} . This initiates a cascade of sector failures, ultimately resulting in 50 sectors ceasing production.

Fig. 10(c) shows an economy where $\kappa_s \sim U(\underline{\kappa}, 10)$, so that even more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors causes the failure of the fragile sectors (now initially 26) and this initiates a cascade of sector failures which ultimately brings down all 100 sectors of the economy.

The discontinuous drops in output caused by fragility, combined with the simple macroeconomic interdependence that we have outlined, come together to form an amplification channel reminiscent, e.g., of Elliott, Golub, and Jackson (2014) and Baqaee (2018). Thus, the implications of those studies apply here: both the cautions regarding the potential severity of knock-on effects, as well as the importance of preventing first failures before they can cascade.

²³Note that in this example, the cascade dynamics is as follows: At step 1, firms in sectors with a κ in the fragile range fail due to an infinitesimal shock to \underline{x} . The initial economy-wide output Y_1 is then decreased to Y_2 and the κ 's are updated using an updating function $K(Y)$ increasing in Y . Only then, are the firms in the surviving sectors allowed re-adjust x_{if} . At step 2, infinitesimal shocks hit again and the firms newly found in the fragile regime fail. This process goes on at each step until no further firm fails, at which point the cascade of failures stops.

²⁴For simplicity, we set $\underline{x} = 0$. An infinitesimal shock to \underline{x} has the effect of causing the firms of sectors in the fragile regime to fail, but does not affect the value of \underline{x} , which remains at 0.

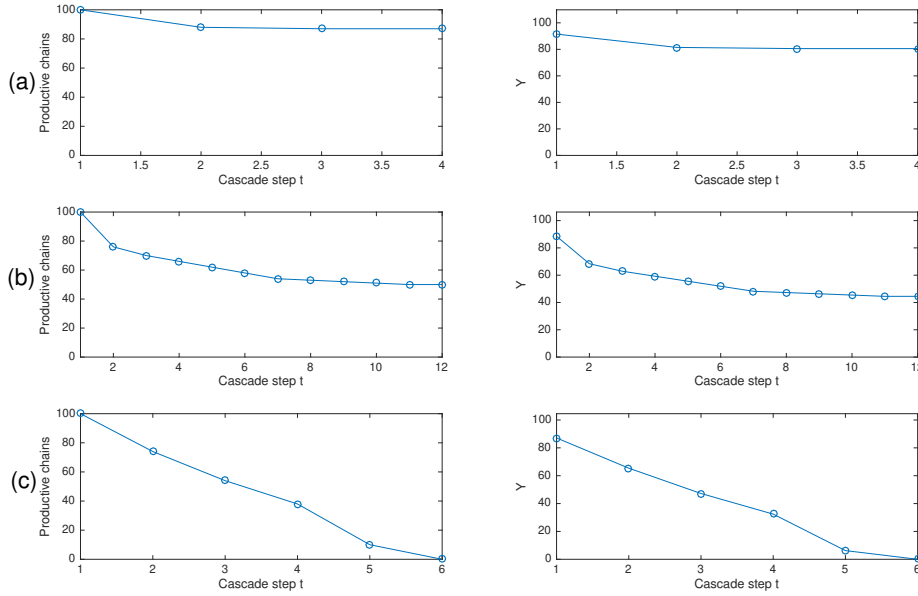


FIGURE 10. Number of sectors that remain productive (left) and economy-wide output Y (right) for each step of a cascade of failures among 100 interdependent sectors. For all sectors: $n = 3$, $m = 5$, $c(x_{if} - \underline{x}) = \frac{0.01}{(1 - (x_{if} - \underline{x}))^2}$, $g(\rho(x)) = 5(1 - \rho(x))$. This yields $\underline{\kappa} = 0.963$ and $\bar{\kappa} = 3.585$. In row (a), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 25)$; in row (b), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 13)$; in row (c), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 10)$.

SA7. PRODUCTION NETWORKS WITH LIMITED DEPTH

The main results of the paper are stated for supply chain depth d sufficiently large. In this section, we numerically explore the shape of the reliability function $\tilde{\rho}(x, d)$ at realistic values of d . We do so while allowing for some systematic heterogeneity (for example, more upstream tiers being simpler).

To motivate the literal modeling of assembly in finitely many tiers, we return to the example of an Airbus A380. This product has 4 million parts. The final assembly in Toulouse, France, consists of six large components coming from five different factories across Europe: three fuselage sections, two wings, and the horizontal tailplane. Each of these factories gets parts from about 1500 companies located in 30 countries²⁵. Each of those companies itself has multiple suppliers, as well as contracts to supply and maintain specialized factory equipment, etc.

As in the main paper consider a depth- d supply tree, but let each firm in tier $t \in \{0, 1, \dots, d\}$ require m_t kinds of inputs and have n_t potential suppliers of each input. Here $t = d$ is the most downstream tier and $t = 0$ is the most upstream tier. The nodes at tier $t = 0$ are functional for sure. As before, we denote by $\tilde{\rho}(x, d)$ the probability of successful production at the most downstream node of a depth- d tree with these properties. This is defined as

$$\tilde{\rho}(x, d) = (1 - (1 - \tilde{\rho}(x, d - 1)x)^{n_d})^{m_d}$$

with $\tilde{\rho}(x, 0) = 1$, since the bottom-tier nodes do not need to obtain inputs.

We see that the expression is recursive and, if unraveled explicitly, would be unwieldy after a number of tiers. However, we know from Definition SA1 and Proposition SA1 that when $m_t = m$ and $n_t = n$ for all t , then for any $x \in [0, 1]$, as d goes to infinity, $\tilde{\rho}(x, d)$ converges to the correspondence $\rho(x)$.

We start with some examples where m_t and n_t are the same throughout the tree. Figure 11 illustrates the successful production probability $\tilde{\rho}(x, d)$ for different finite depths d and how quickly it converges to the correspondence $\rho(x)$.

²⁵Source: “FOCUS: The extraordinary A380 supply chain”. *Logistics Middle East*. Retrieved on 28 may, 2019 from <https://www.logisticsmiddleeast.com/article-13803-focus-the-extraordinary-a380-supply-chain>

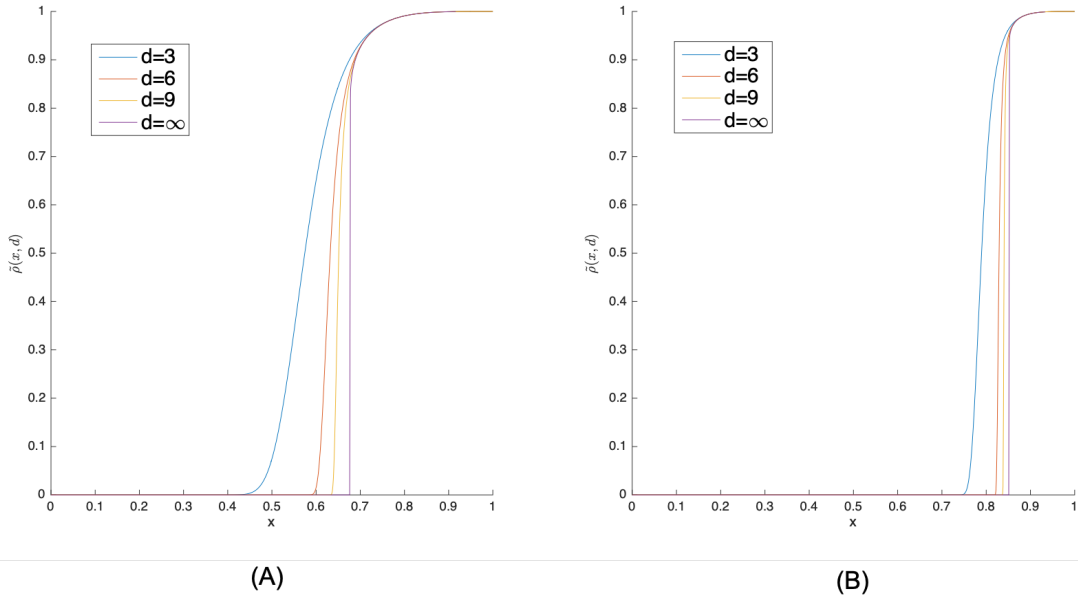


FIGURE 11. Successful production probability $\tilde{\rho}(x, d)$ for different finite numbers of tiers d . In panel (A), $m = 5$ and $n = 4$. In panel (B), $m = 40$ and $n = 4$.

In panel (A), we see that $\tilde{\rho}(x, d)$ exhibits a sharp transition for a depth as small as $d = 3$. The red curve ($d = 6$) shows that when the investment level x drops from 0.66 to 0.61, or about 7 percent, production probability $\tilde{\rho}(x, 6)$ drops from 0.8 to 0.1. (Thus $\tilde{\rho}(x, 6)$ achieves a slope of at least 14.) In panel (B), we see that increasing product complexity (by increasing m to 40) causes $\tilde{\rho}(x, d)$ to lie quite close to $\rho(x)$. This illustrates how complementarities between inputs play a key role in driving this sharp transition in the probability of successful production. Note that $m = 40$ is not an exaggerated number in reality. In the Airbus example described earlier, many components would exhibit such a level of complexity.

However, a complexity number like $m = 40$ will not occur everywhere throughout the supply network. Indeed, and more generally, one might ask whether the regularity in the production tree is responsible for the sharpness of the transition. To investigate this possibility, in Figure 12 we plot $\tilde{\rho}(x, d)$ for a supply tree with irregular complexity, where different tiers may have different values of m_t . Here we construct 4 trees whose complexity increases with d . The first tree has $d = 3$ and $m_1 = 2$, $m_2 = 6$ and $m_3 = 10$. The second tree has $d = 6$, $m_t = 2$ for $t = 1, 2$, $m_t = 6$, for $t = 3, 4$ and $m_t = 10$, for $t = 5, 6$. The third tree has $d = 9$, $m_t = 2$ for $t = 1, 2, 3$, $m_t = 6$, for $t = 4, 5, 6$ and $m_t = 10$, for $t = 7, 8, 9$. Finally, the fourth tree has large depth (here $d = 999$), $m_t = 2$ for t below the first tercile, $m_t = 6$ for t between the first and the second terciles and $m_t = 10$ for t above the second tercile. We see that trees of moderate depth once again exhibit a sharp transition in their probability of successful production. This feature is thus not at all dependent upon the regularity of the trees.

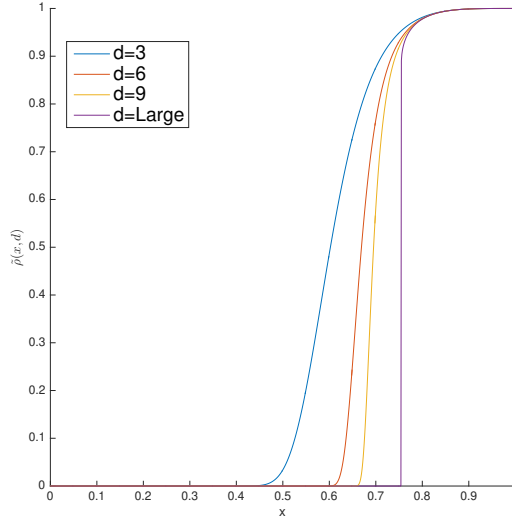


FIGURE 12. Successful production probability $\tilde{\rho}(x, d)$ for different finite numbers of tiers d , but where different tiers may have different complexity m .

SA8. HETEROGENEITY

There is a finite set of products, \mathcal{I} . Each product $i \in \mathcal{I}$ is associated with a *product complexity* m_i and a finite set of inputs $I(i) \subseteq \mathcal{I}$ of cardinality m_i . Thus, the number of inputs required can be different for different products. For each product i and input $j \in I(i)$, there is a number n_{ij} of potential suppliers of product j that each firm has; thus n_{ij} replaces the single multisourcing parameter n . For each pair $i, j \in \mathcal{I}$, there is a relationship strength x_{ij} such that every link where product i sources product j has a probability x_{ij} of being operational.

We generalize our basic model of depth by assigning each firm a generalized depth. Let the set of depth types be \mathcal{D} , indexed by the nonnegative integers, with a typical element denoted δ . The lowest depth type firms, with $\delta = 0$, require no specific sourcing. Any higher depth type firm sources only from depth types lower than it. We then construct the potential supply network as follows: Each firm in industry i has a depth type $\delta_{if} \in \mathcal{D}$. There is a distribution $\mu_i \in \Delta(\mathcal{D})$ of depth types in product i . For every $i \in \mathcal{I}$, $j \in I(i)$, and $\delta \in \mathcal{D}$ with $\delta > 0$ and $\mu_i(\delta) > 0$, there is a measure $\mu_{ij, \delta} \in \Delta(\mathcal{D} \cap [0, \delta])$ which is the distribution of depths of product j when i sources from j , assuming $n_{ij} > 0$. The distribution of the depths of if 's suppliers can be different in different inputs j . A depth type has no required inputs if and only if it is of depth zero. We assume that draws of $\delta_{jf'}$ are independent across different suppliers jf' (just for simplicity). We assume that if there is a positive measure of firms of depth δ in product i and some j, δ' for which $\mu_{ij, \delta}(\delta') > 0$ (i.e., there is a positive probability of depth- δ firms in i matching with depth- δ' firms in j) then $\mu_j(\delta') > 0$.²⁶

We also need to impose conditions to make the finite depth model converge in a suitable sense to an infinite model. Define a profile

$$\boldsymbol{\mu}(\tau) = [(\mu_i(\tau))_{i \in \mathcal{I}}, (\mu_{ij, \delta}(\tau))_{i \in \mathcal{I}, j \in I(i), \delta \in \mathcal{D}}]$$

of depth distributions to be parameterized by τ as in the basic model. We impose a condition that captures that, as $\tau \rightarrow \infty$, chains become deeper.

Assumption SA1.

²⁶We do not need to impose any other accounting identities, since the average number of in-links to any variety is arbitrary and does not play a role in our results. An equivalent way of looking at the structure just described is that supply networks are (as before) directed acyclic graphs—see Tintelnot et al. (2018) for a similar model. This implies that there are some firms that require no specific sourcing. We assign these firms depth $\delta = 0$. Removing these firms from the network, there must then be some remaining firms that require no specific sourcing from the other remaining firms. These are depth $\delta = 1$ firms. Proceeding iteratively we assign all firms a depth. This allows a firm to source from other firms with different depths but implies that a depth k firm source only from firms with depth less than k .

- (1) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\delta, \delta' \in D$ such that $\delta' > \delta$, the distribution $\mu_{ij, \delta'}$ first-order stochastically dominates $\mu_{ij, \delta}$.
- (2) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\bar{\delta} \in D$, the mass placed on $[0, \bar{\delta}]$ by $\mu_{ij, \delta}(\tau)$ tends to 0 as $\delta \rightarrow \infty$.
- (3) For any $i \in \mathcal{I}$, any $j \in I(i)$, and any $\bar{\delta} \in D$, the mass placed on $[0, \bar{\delta}]$ by $\mu_i(\tau)$ tends to 0 as $\tau \rightarrow \infty$.

Now we can iteratively define reliabilities for this model. Let $\tilde{\rho}_{i, \delta}(x_{ij}) = 1$ if $\delta = 0$ and, inductively, for $\delta > 0$,

$$\tilde{\rho}_i(\delta) = \prod_{j \in I(i)} \mathbb{E}_{\delta' \sim \mu_{ij, \delta}} [1 - (1 - \tilde{\rho}_j(\delta') x_{ij})^{n_{ij}}]. \quad (\text{SA-30})$$

In the expectation, we are summing over different values of δ' , weighted by probabilities $\mu_{ij, \delta}(\delta')$. Analogously to the main model, we define

$$\rho_i = \mathbb{E}_{\delta \sim \mu_{ij}} \tilde{\rho}_i(\delta), \quad (\text{SA-31})$$

and we let $\boldsymbol{\rho}(\mathbf{x}, \boldsymbol{\mu})$ denote the (multidimensional) reliability function thus constructed.

Again paralleling the main model, we will now study a function that corresponds to the limit of the system (SA-30) as $\tau \rightarrow \infty$. We will use the notation r_i for the fraction of firms in product i functioning, and let \mathbf{r} denote a vector of all such reliabilities. Define $\mathcal{R}(\mathbf{r})$ by

$$[\mathcal{R}(\mathbf{r})]_i = \prod_{j \in I(i)} [1 - (1 - r_j x_{ij}(\xi))^{n_{ij}}]. \quad (\text{SA-32})$$

LEMMA SA16. There is a pointwise-largest fixed-point of the function $\mathcal{R}(\mathbf{r})$, denoted by $\hat{\boldsymbol{\rho}}(x)$. Moreover, $\boldsymbol{\rho}(x, \tau)$ converges pointwise to $\hat{\boldsymbol{\rho}}(x)$ as $\tau \rightarrow \infty$.

Proof. By Tarski's theorem, since \mathcal{R} is monotone, it has a pointwise largest fixed point.

By inductively using Assumption SA1(1), it can be shown that $\tilde{\rho}(\delta)$ is pointwise decreasing in δ . This monotone sequence must have a limit point, and by applying Assumption SA1(2) we can see that it must be a fixed point of \mathcal{R} . Moreover, the argument of Echenique (2005, Section 4) shows that it is the largest fixed point. Finally, Assumption SA1(3) guarantees that $\boldsymbol{\rho}(x)$ converges to this fixed point as well. \square

We now prove that a sharp transition in production probability arises in the heterogeneous setting (Proposition SA2) by analyzing the $\hat{\boldsymbol{\rho}}(x)$ defined in Lemma SA16. We then present two examples exhibiting the fragility and weakest link properties (Subsection SA8.3). Finally, we present numerical examples and show that fragility is compatible with endogenous investment (Subsection SA8.3.5).

SA8.1. Generalization of the sharp transition in the heterogeneous case. To show the existence of a discontinuity in $\hat{\boldsymbol{\rho}}$, we study the function along a single curve in \mathbf{x} space. To this end we introduce a single parameter ξ and define $x_{if, j} = x_{ij}(\xi)$, where x_{ij} are strictly increasing, differentiable, surjective functions $[0, 1] \rightarrow [0, 1]$ with $x_{ij}(1) = 1$.

PROPOSITION SA2. Suppose that for all products i , the complexity m_i is at least 2. Moreover, suppose whenever $j \in I(i)$, the number n_{ij} of potential suppliers for each firm is at least 1. For any product i , the measure of the set of functional firms $\bar{\mathcal{F}}_i$, denoted by $\rho_i(\xi)$, is a nondecreasing function with the following properties.

- (1) There is a number ξ_{crit} and a vector $\mathbf{r}_{\text{crit}} > 0$ such that $\boldsymbol{\rho}$ has a discontinuity at ξ_{crit} , where it jumps from 0 to \mathbf{r}_{crit} and is strictly increasing in each component after that.
- (2) If $n_{ij} = 1$ for all i and j , we have that $\xi_{\text{crit}} = 1$.
- (3) If $\xi_{\text{crit}} < 1$, then as ξ approaches ξ_{crit} from above, the derivative $\rho'_i(x)$ tends to ∞ in some component.

The idea of this result is simple, and generalizes the graphical intuition of Figure 4 in Section IIC of the paper. For any $\mathbf{r} \in [0, 1]^{|\mathcal{I}|}$, define $\mathcal{R}_\xi(\mathbf{r})$ to be the probability, under the parameter ξ , that a

producer of product i is functional given that the reliability vector for producers of other products is given by \mathbf{r} . This can be written, using (SA-32), as

$$[\mathcal{R}_\xi(\mathbf{r})]_i = \prod_{j \in I(i)} [1 - (1 - r_{jx_{ij}}(\xi))^{n_{ij}}].$$

Near 0, the map $\mathcal{R}_\xi : [0, 1]^n \rightarrow [0, 1]^n$ is bounded above by a quadratic function (as a consequence of $m_i \geq 2$ for all i). Therefore it cannot have any fixed points near 0. Thus, analogously to Figure 4 in the main paper, fixed points disappear abruptly as ξ is reduced past a critical value ξ_{crit} .

Proof of Proposition SA2. For any $\mathbf{r} \in [0, 1]^{|Z|}$, define $\mathcal{R}_\xi(\mathbf{r})$ to be the probability, under the parameter ξ , that a producer of product i is functional given that the reliability vector for all products is given by \mathbf{r} . This can be written explicitly:

$$[\mathcal{R}_\xi(\mathbf{r})]_i = \prod_{j \in I(i)} [1 - (1 - r_{jx_{ij}}(\xi))^{n_{ij}}].$$

Let $\hat{\rho}(\xi)$ be the elementwise largest fixed point of \mathcal{R}_ξ , which exists and corresponds to the mass of functional firms by the same argument as in Lemma SA1.

(1) It is clear that $\hat{\rho}(1) = \mathbf{1}$.

(2) Next, there is an $\epsilon > 0$ such that if $\|\mathbf{r}\| < \epsilon$, then for all ξ , the function $\mathcal{R}_\xi(\mathbf{r}) < \mathbf{r}$ elementwise. So there are no fixed points near $\mathbf{0}$.

(3) For small enough ξ , the function \mathcal{R}_ξ is uniformly small, so $\hat{\rho}(\xi) = \mathbf{0}$.

These facts together imply that $\hat{\rho}$ has a discontinuity where it jumps up from $\mathbf{0}$. Let ξ_{crit} be the infimum of the ξ where $\hat{\rho}(\xi) \neq \mathbf{0}$.

Define

$$\Gamma(\mathcal{R}_\xi(\mathbf{r})) = \{(\mathbf{r}, \mathcal{R}_\xi(\mathbf{r})) : \mathbf{r} \in [0, 1]^{|Z|}\}$$

to be the graph of the function. What we have just said corresponds to the fact that this graph intersects the diagonal when $\xi = \xi_{\text{crit}}$, but not for values $\xi < \xi_{\text{crit}}$. Suppose now, toward a contradiction, that the derivative $\delta(\xi)$ of $\hat{\rho}(\xi)$ is bounded in every coordinate as $\xi \downarrow \xi_{\text{crit}}$. Then, passing to a convergent subsequence and using the smoothness of \mathcal{R} , we find that the derivative $\delta(\xi_{\text{crit}})$ of $\hat{\rho}$ is well-defined at $\xi = \xi_{\text{crit}}$. But that contradicts our earlier deduction that $\hat{\rho}$ is discontinuous at ξ_{crit} . \square

This result shows that there are profiles \mathbf{x} of relationship strengths where small reductions in strength cause a discontinuous drop in reliability (in the limit model). We can define a correspondence $\rho(\mathbf{x})$ which is limit of the reliability functions²⁷ $\rho(\mathbf{x}, \mu(\tau))$ as $\tau \rightarrow \infty$. This is equal to $\hat{\rho}(\mathbf{x})$ except at the discontinuities, where $\hat{\rho}(\mathbf{x})$ is multi-valued. Paralleling the case of homogeneous firms, this implies the potential for endogenous fragility. We show in Section SA8.3, via a numerical example, that this potential is realized: fragile outcomes do indeed obtain (for an open set of parameters).

SA8.2. Proof of Proposition 5 (Weakest link). Here we prove Proposition 5 in the main text.

Part (i): Let \mathcal{P} be a directed path of length T from node T to node i and denote product i by 1. Consider a shock that decreases all relationship strengths, $\mathbf{x}' < \mathbf{x}$ elementwise, and thus makes $r'_1 = 0$ by definition of i being critical, where the ‘‘prime’’ notation denotes the quantity after the shock.

If any product $t + 1$ sources input t and $r'_t = 0$, we have that

$$\begin{aligned} r'_{t+1} &= \prod_{l \in I(t+1)} (1 - (1 - x'_{t+1,l} r'_l)^{n_{t+1,l}}) \\ &= 0 \end{aligned}$$

since $t \in I(t + 1)$.

Since $r'_1 = 0$, it then follows by induction that the production of all products $t \in \mathcal{P}$ will fail.

Part (ii): This follows immediately from Part (i) and the fact that i and j are in the same strongly connected component \mathcal{I}^{SC} if and only if there is a directed path in the product dependence graph from each product to the other. \square

²⁷Recall (SA-31).

SA8.2.1. *Endogenous investment.* So far we have discussed only the mechanics of the model. Now we explicitly model investment. For a firm if and an input $j \in I(i)$, the cost of effort is $c_{ij}(x_{if,j} - \underline{x}_{ij})$. The gross profit conditional on producing product i is $g_i(r_i)$. Here g_i is a product-specific function (which can capture many different features of different product markets that affect their profitabilities) and r_i is the reliability of producers of product i . We work here with a reduced-form specification of gross profits, and assume that all functions satisfy the same assumptions as their counterparts in the homogeneous model. As before, shocks are modeled as reducing \underline{x}_{ij} after investments have been made.

Investment decisions are made simultaneously by firms, knowing their product type i but ex ante of depth realizations. Our solution concept is symmetric undominated equilibrium.

A full theoretical analysis of this model is beyond our scope, but in the following sections we show numerically that fragility of the type we have identified is consistent with equilibrium investment, and illustrate the other phenomena that arise in the model with heterogeneities.

SA8.3. Examples with heterogeneities.

SA8.3.1. *The mechanics of production.* There are seven products. Only product a is used as an input into its own production. Products a, b, c and d all use inputs from each other; products e, f , and g also all use inputs from each other, but also require product a as an input. Figure 13 shows the input dependencies between these products. There are two strongly connected components. Products a, b, c , and d form a strongly connected component while e, f , and g form another strongly connected component. Products a and b have three potential suppliers for each of their required inputs, while all other products have only two potential suppliers for each of their required inputs.

SA8.3.2. *Implications for criticality.* We can begin with some general observations that do not depend on any primitives beyond what has been described. By Proposition 5(ii), products within a strongly connected component must be all critical or all non-critical. By Proposition 5(i), if a given product is critical, all products that source this product directly or indirectly must also be critical. Combining these results, there are just three possibilities: no products are critical; only products e, f, g are critical; all products are critical. Note that it is impossible for products a, b, c , and d to be critical while products e, f , and g are non-critical. The reason is that if production of product a fails, then the e, f and g producers will be not be able to source all the inputs they need, and so will also be unable to produce. On the other hand, note that production of the a, b, c and d products does not require sourcing any of the e, f, g products, so we can have the e, f, g products be critical while the a, b, c, d products are not.

SA8.3.3. *An outcome at which a strict subset of firms is critical: Mechanics.* The system $\mathbf{r} = \mathcal{R}(\mathbf{r})$ can be written as, for all i

$$r_i = \prod_{j \in I(i)} (1 - (1 - x_{ij}r_j)^{n_{ij}}).$$

Here $I(i)$ is the neighborhood of i on the product dependency graph, with $|I(i)| = m_i$ (the complexity of production for product i), and n_{ij} is the number of potential suppliers a producer of product i has for input j (i.e. the potential level of multisourcing by producers of product i for input j).

We will now exhibit a point (\mathbf{x}, \mathbf{r}) at which products $\{a, b, c, d\}$ are critical in the limit model (and thus for large enough depths).²⁸ An entry x_{ij} in the following matrix represents the strength chosen by a producer of product i in a relationship sourcing input j .

$$\mathbf{x} = \begin{matrix} & \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{matrix} & \left[\begin{array}{ccccccc} 0.8873 & 0.8872 & 0.9315 & 0.9385 & 0 & 0 & 0 \\ 0.8773 & 0 & 0.9204 & 0.9272 & 0 & 0 & 0 \\ 0.8673 & 0.8672 & 0 & 0.9084 & 0 & 0 & 0 \\ 0.8573 & 0.8572 & 0.8915 & 0 & 0 & 0 & 0 \\ 0.7573 & 0 & 0 & 0 & 0 & 0.9726 & 0.9783 \\ 0.7473 & 0 & 0 & 0 & 0.9464 & 0 & 0.9572 \\ 0.7373 & 0 & 0 & 0 & 0.9265 & 0.9317 & 0 \end{array} \right] \end{matrix}. \quad (\text{SA-33})$$

The product reliabilities \mathbf{r} are as follows:

$$\mathbf{r} = [0.9926, 0.9928, 0.9387, 0.9307, 0.5384, 0.5262, 0.5145]. \quad (\text{SA-34})$$

²⁸At the end of SA8.3 we describe how this example was generated.

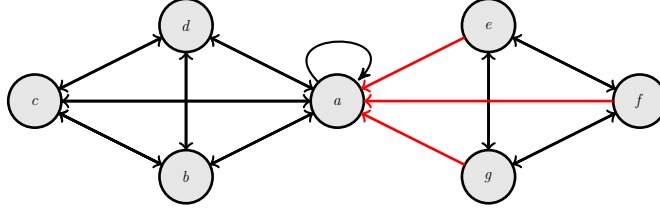


FIGURE 13. Supply dependencies: Bidirectional arrows represent reciprocated supply dependencies in which both products require inputs from each other. A red one-directional arrow from one product to another means that the product at the origin of the arrow uses as an input the product at the end of the arrow (e.g. product e requires product a as an input, but not the other way around). Product a also depends on itself, reflected by the loop. Note that a, b, c, d form a strongly connected component while e, f, g form another strongly connected component.

For this configuration, products e, f and g are critical, while a, b, c , and d are noncritical. To see this, we consider a small unanticipated shock to the relationship strengths \mathbf{x} (e.g., we reduce all entries of \mathbf{x} by a small δ). For firms producing products a, b, c or d , the impact of this is minor. The probability of successful production for those products (r_i) only drops continuously and for a small shock the change will be small. On the other hand, the small shock is sufficient for the output of the firms producing products e, f and g to collapse to 0. This can be seen by iteratively applying \mathcal{R} after the small shock.

SA8.3.4. *An outcome at which a strict subset of firms is critical: Endogenous investment.* We now specify (heterogeneous) cost and profit functions that support the outcome described above as an equilibrium. Specifically, we let the gross profit function of a firm producing product i be

$$g_i(r_i) = \alpha_i(1 - r_i),$$

where

$$\alpha = [2857.4, 2456.1, 53.5, 43.9, 5.8, 5.5, 5.1].$$

We let the cost of a producer of product i from investing in supplier relationships with producers of product j be

$$c_{ij}(x_{ij} - \underline{x}_{ij}) = \frac{1}{2}\gamma_{ij}x_{ij}^2.$$

For simplicity, we have set $\underline{x}_{ij} = 0$ and we set $\gamma_{ij} = 1$ for all product pairs ij .

As we have mentioned, we consider symmetric equilibria, in the sense that all producers of product i invest the same amount sourcing a given input j . As we are interested in these symmetric investment choices, we dispense with the subscript f when we consider investments. In such an equilibrium the profit of a producer of product i is given by

$$\Pi_i = g_i(r_i)r_i - \frac{1}{2} \sum_{j \in I(i)} \gamma_{ij}x_{ij}^2, \quad (\text{SA-35})$$

From these specifications, it can be checked numerically that the x_{ij} in (SA-33) maximize the profit function given the levels of reliability in (SA-34). Our discussion below will show how the parameters, by construction, satisfy the first-order conditions of agents' optimization problems.

SA8.3.5. *Finding a critical outcome.* We now describe how we reduced the search for an example to a simple numerical problem.

The marginal benefit a producer of product i receives from investing in its relationships with suppliers of input j is

$$MB_{ij} = g_i(r_i) \prod_{l \in I(i), l \neq j} (1 - (1 - x_{il}r_l)^{n_{il}}) n_{ij} (1 - x_{ij}r_j)^{n_{ij}-1} r_j. \quad (\text{SA-36})$$

The marginal cost for a producer of product i investing in a relationship with a supplier of input j is

$$MC_{ij} = x_{ij}. \quad (\text{SA-37})$$

We look for $|\mathcal{I}| \times |\mathcal{I}|$ matrix \mathbf{x} (i.e., the matrix containing the investment profiles for all products), with entries x_{ij} satisfying $MB_{ij} = MC_{ij}$, along with parameters α supporting this outcome. We will view the α as free parameters to be chosen in our search.

Fix a conjectured $\check{r} \in \mathbb{R}^{|\mathcal{I}|}$ (we begin with the vector of ones) and arbitrary $x_{i1} \in (0, 1)$ for all $i \in \mathcal{I}$. The value of x_{i1} that equates the marginal benefits and marginal costs for a producer of product i 's investment into sourcing product 1 is increasing in $g_i(r_i) = \alpha_i(1 - r_i)$, and covers the $[0, 1]$ interval as α_i varies. We can thus choose the free parameter α_i to achieve the desired $x_{i1} \in (0, 1)$. We claim that the choice of x_{i1} for all i and \check{r} then pins down the value of x_{ij} for all $j \neq 1$ in any best-response profile, as follows. We must have

$$\frac{MB_{ij}}{MB_{i1}} = \frac{MC_{ij}}{MC_{i1}}, \quad \text{for all } i, j$$

which can be expressed as

$$\frac{g_i(\check{r}_i) \prod_{l \in \mathcal{I}(i), l \neq j} (1 - (1 - x_{il}\check{r}_l)^{n_{il}}) n_{ij} (1 - x_{ij}\check{r}_j)^{n_{ij}-1} \check{r}_j}{g_i(\check{r}_i) \prod_{l \in \mathcal{I}(i), l \neq 1} (1 - (1 - x_{il}\check{r}_l)^{n_{il}}) n_{i1} (1 - x_{i1}\check{r}_1)^{n_{i1}-1} \check{r}_1} = \frac{x_{ij}}{x_{i1}},$$

and reduces to

$$\frac{(1 - x_{ij}\check{r}_j)^{n_{ij}-1}}{(1 - (1 - x_{ij}\check{r}_j)^{n_{ij}})} \frac{n_{ij}}{x_{ij}} = \frac{n_{i1}}{x_{i1}} \frac{\check{r}_1}{\check{r}_j} \frac{(1 - x_{i1}\check{r}_1)^{n_{i1}-1}}{(1 - (1 - x_{i1}\check{r}_1)^{n_{i1}})}. \quad (\text{SA-38})$$

The left-hand side is decreasing in x_{ij} while the right-hand side is given, so there can be only one solution x_{ij} satisfying the above.

Setting expressions (SA-36) and (SA-37) equal to each other implies a value for $g_i(r_i)$ for all $i \neq 1$. Recall that $g_i(r_i) = \alpha_i(1 - r_i)$, and so depends on α_i . Thus, for a given value of $g_i(r_i)$ we set

$$\alpha_i = \frac{g_i(\check{r}_i)}{1 - \check{r}_i}.$$

This, along with our calculations above, ensures that all firms satisfy their first-order conditions for investment at reliabilities \check{r} .

Now we describe the procedure that we use to search for an equilibrium where some products are critical.

- Initialize $(x_{i1})_{i \in \mathcal{I}}$ to values just smaller than 1 and decrease these values incrementally by a small amount along an arbitrary strictly decreasing curve.²⁹
- For each $(x_{i1})_{i \in \mathcal{I}}$, repeat ...
 - calculate the entire \mathbf{x} matrix using the above procedure;
 - update the conjectured reliabilities by updating $\check{r} \leftarrow \widehat{\rho}(\mathbf{x})$.
 ... until $\|\widehat{\rho}(\mathbf{x}) - \check{r}\|$ is within a desired tolerance of 0.
- Move on to the next $(x_{i1})_{i \in \mathcal{I}}$.

We continue until the probability of successful production decreases to 0 for one of the products i . We then look at the $(x_{i1})_{i \in \mathcal{I}}$ just before this has happened. This gives us a value of \mathbf{x} (i.e., the investment profiles for all products) such that, for any given shock magnitude, at least one product has reliability 0 after the shock.

As we have discussed, the equations above give us parameters such that the first-order conditions for optimality hold. We can then check for global optimality numerically.

The above algorithm helped us find $(\alpha_{\text{crit}}, \mathbf{x}_{\text{crit}}, \mathbf{r}_{\text{crit}})$ together constituting an equilibrium in the infinite-depth model, corresponding to $\widehat{\rho}$ in Section SA8.1. In the finite-depth version of the model, the only difference is that $\widehat{\rho}(\mathbf{x})$ should be replaced by $\rho(\mathbf{x}, \boldsymbol{\mu})$ for deep³⁰ $\boldsymbol{\mu}$. On the part of $\widehat{\rho}(\mathbf{x})$ visited by the algorithm, the finite-depth reliability and best-response functions converge to $\widehat{\rho}(\mathbf{x})$ pointwise (and, by a compactness argument, therefore uniformly). It follows that the computations of the finite-depth algorithm converge uniformly to the ones defined here, under a standard nonsingularity condition at the equilibrium which can be checked numerically.³¹

²⁹We chose the curve to find an example where one component of the network becomes critical before the whole network becomes critical.

³⁰We can each product's suppliers' depths to be drawn from a Poisson distribution with mean τ just for concreteness and let $\tau \rightarrow \infty$.

³¹Intuitively, paralleling the illustration of Theorem 1, the condition is that the equilibrium is not at a point where the two curves are tangent.

SA8.3.6. *A different set of critical products.* As we have mentioned, there are essentially two types of equilibria with fragile firms. If a firm in the set $\{e, f, g\}$ becomes critical first, then all firms in this set simultaneously become critical, but not necessarily the others. This is the case we have seen above.

When a firm in the set $\{a, b, c, d\}$ becomes fragile first, a shock to any one of $\{a, b, c, d\}$ that reduces the reliability of sourcing an input is sufficient for the probability of successful production of *all* firms to fall to 0.

To illustrate the latter possibility, We adjust the configuration of the previous example by letting the vector of product profitabilities be

$$\alpha = [32.46, 45.37, 8.52, 9.24, 21.78, 24.62, 28.00].$$

Everything else remains the same as before.

The equilibrium investment levels are reported in the matrix \mathbf{x} below.

$$\mathbf{x} = \begin{bmatrix} 0.7965 & 0.7792 & 0.8735 & 0.8663 & 0 & 0 & 0 \\ 0.8065 & 0 & 0.8859 & 0.8785 & 0 & 0 & 0 \\ 0.8165 & 0.8029 & 0 & 0.8681 & 0 & 0 & 0 \\ 0.8265 & 0.8124 & 0.8855 & 0 & 0 & 0 & 0 \\ 0.8965 & 0 & 0 & 0 & 0 & 0.8947 & 0.8894 \\ 0.9065 & 0 & 0 & 0 & 0.9103 & 0 & 0.8992 \\ 0.9165 & 0 & 0 & 0 & 0.9204 & 0.9146 & 0 \end{bmatrix}.$$

The reliabilities are as follows

$$\mathbf{r} = [0.8837, 0.9132, 0.7653, 0.7756, 0.8778, 0.8865, 0.8951].$$

This example is constructed analogously to the previous one, just tuning the starting point to reach a different critical outcome. Indeed, at these parameter values production of all products is now critical. Following a small shock to relationship strengths \mathbf{x} (again, we reduce all relationship strengths a little) reliability of all products collapses to 0.

SA9. INTERPRETATION OF INVESTMENT

SA9.1. **Effort on both the extensive and intensive margins.** This section supports the claims made in Section VC of the paper that our model is easily extended to allow firms to make separate multi-sourcing effort choices on the intensive margin (quality of relationships) and the extensive margin (finding potential suppliers).

Suppose a firm if chooses efforts $\widehat{e}_{if} \geq 0$ on the extensive margin and effort $\widetilde{e}_{if} \geq 0$ on the intensive margin, and suppose that $x_{if} = h(\widehat{e}_{if}, \widetilde{e}_{if})$. Let the cost of investment be a function of $\widehat{e}_{if} + \widetilde{e}_{if}$ instead of y_{if} . This firm problem can be broken down into choosing an overall effort level $e_{if} = \widehat{e}_{if} + \widetilde{e}_{if}$ and then a share of this effort level allocated to the intensive margin, with the remaining share allocated to the extensive margin. Fixing an effort level e , a firm will choose $\widehat{e}_{if} \in [0, e]$, with $\widetilde{e}_{if} = e - \widehat{e}_{if}$, to maximize x_{if} . Let $\widehat{e}_{if}^*(e)$ and $\widetilde{e}_{if}^*(e) = e - \widehat{e}_{if}^*(e)$ denote the allocation of effort across the intensive and extensive margins that maximizes x_{if} given overall effort e . Given these choices, define $h^*(e) := h(\widehat{e}_{if}^*(e), \widetilde{e}_{if}^*(e))$. As h^* is strictly increasing in e , choosing e is then equivalent to choosing x_{if} directly, with a cost of effort equal to $c(h^{*-1}(e))$. Thus, as long as the cost function $\widehat{c}(e) := c(h^{*-1}(e))$ continues to satisfy our maintained assumptions on c , everything goes through unaffected.

SA9.2. **A richer extensive margin model.** In the previous subsection we gave an extensive margin search effort interpretation of x_{if} . In some ways this interpretation was restrictive. Specifically, it required there to be exactly n suppliers capable for supplying the input and that each such supplier be found independently with probability x_{if} . This alternative interpretation is a minimal departure from the intensive margin interpretation, which is why we gave it. However, it is also possible, through a change of variables, to see that our model encompasses a more general and standard search interpretation.

Fixing the environment a firms faces, specifically the probability other firms successfully produce $r > 0$ and a parameter n that will index the ease of search, suppose we let each firm if choose directly the probability that, through search, it finds an input of given type. When $r = 0$ we suppose that all search is futile and that firms necessarily choose $\widehat{x}_{if} = 0$. Denote the probability firm if finds a supplier of a given input type by \widehat{x}_{if} . Conditional on finding an input, we let it be successfully sourced with

probability 1 so all frictions occur through the search process. Implicitly, obtaining a probability \hat{x}_{if} requires search effort, and we suppose that cost of achieving probability \hat{x}_{if} is $\hat{c}(\hat{x})$, where \hat{c} is a strictly increasing function with $\hat{c}(0) = 0$.

We suppose firms choose \hat{x}_{if} taking the environment as given. In particular, firms take as given the probability that suppliers of the inputs they require successfully produce. When many potential suppliers of an input produce successfully we let it be relatively easy to find one, and if none of these suppliers produce successfully then it is impossible to find one. In addition, the parameter n shifts how easy it is to find a supplier.

Given this set up we can let the probability of finding a supplier have the functional form $\hat{x} := 1 - (1 - x_{if}r)^n$, and the cost of achieving this probability be given by $\hat{c}(\hat{x}) := c \left(\frac{1 - (1 - \hat{x})^{1/n}}{r} \right)$. Although these functional form assumptions might seem restrictive, we still have freedom to use any function c satisfying our maintained assumptions. This degree of freedom is enough for the model to be quite general as all that matters is the size of the benefits of search effort relative to its cost, and not the absolute magnitudes. Further, these functional form assumptions satisfy all the desiderata we set out above. As $1 - (1 - x_{if}r)^n$ is the key probability throughout our analysis, all our results then go through with this interpretation.

REFERENCES

- BAQAEI, D. R. (2018): “Cascading Failures in Production Networks,” *Econometrica*, 86, 1819–1838.
- BOEHM, C. E., A. FLAAEN, AND N. PANDALAI-NAYAR (2019): “Input Linkages and the Transmission of Shocks: Firm-Level Evidence from the 2011 Tōhoku Earthquake,” *Review of Economics and Statistics*, 101, 60–75.
- ECHENIQUE, F. (2005): “A short and constructive proof of Tarski’s fixed-point theorem,” *International Journal of Game Theory*, 33, 215–218.
- ELLIOTT, M., B. GOLUB, AND M. O. JACKSON (2014): “Financial Networks and Contagion,” *American Economic Review*, 104, 3115–53.
- KIRMAN, A. P. AND N. J. VRIEND (2000): “Learning to be Loyal. A Study of the Marseille Fish Market,” in *Interaction and Market structure*, Springer, 33–56.
- TINTELNOT, F., A. K. KIKKAWA, M. MOGSTAD, AND E. DHYNE (2018): “Trade and Domestic Production Networks,” Working Paper 25120, National Bureau of Economic Research.
- UZZI, B. (1997): “Social Structure and Competition in Interfirm Networks: The Paradox of Embeddedness,” *Administrative Science Quarterly*, 35–67.