INCENTIVE DESIGN WITH SPILLOVERS

KRISHNA DASARATHA, BENJAMIN GOLUB, AND ANANT SHAH

Abstract. A principal uses payments conditioned on stochastic outcomes of a team project to elicit costly effort from the team members. We develop a multi-agent generalization of a classic first-order approach to contract optimization by leveraging methods from network games. The main results characterize the optimal allocation of incentive pay across agents and outcomes. Incentive optimality requires equalizing, across agents, a product of (i) individual productivity (ii) organizational centrality and (iii) responsiveness to monetary incentives.

Dasaratha: krishnadasaratha@gmail.com, Boston University. Golub: ben.golub@gmail.com, Northwestern University. Shah: anantshah2026@u.northwestern.edu, Northwestern University. We thank Yann Calvó López, Yu-Chi Hsieh, Vitalii Tubdenov for excellent research assistance. We are grateful to Drew Fudenberg and his reading group for a very valuable discussion, and also for helpful comments from (in random order) to Erik Madsen, Ilya Segal, George Mailath, Aravindan Vijayaraghavan, Stephen Morris, Omer Tamuz, Marina Halac, Thomas Steinke, Marzena Rostek, Daron Ravid, Jose Betancourt Valencia, Roberto Corrao, Jacopo Perego, Juan Ortner, Alex Wolitzky, Jason Hartline, Evan Sadler, Adam Szeidl, Sam Wycherley, Michael Ostrovsky, Elliot Lipnowski, Yifan Dai, Melika Liporace, Piero Gottardi, and Michael Powell, as well as many seminar and conference participants.

1. Introduction

A popular method of motivating members of a team is giving them performance incentives that depend on jointly achieved outcomes. Examples of such incentives include startup executives receiving firm stock and a marketing team receiving bonuses for achieving a sales target. How should such incentive schemes be designed and how should they take into account the team's production function?

We examine these questions in a simple non-parametric model of a team working on a joint project. Each member chooses how much costly effort to exert. These actions jointly determine a real-valued *team performance*—for example, the quality of a product—according to a nice, increasing function of the efforts, which may entail interactions such as complementarities among agents' efforts. Any performance level determines a probability distribution over observable project *outcomes*. For example, the outcome may be the revenue from a project, which is stochastically increasing in non-contractible project quality. The uncertainty reflects factors outside the team's control, such as competing product releases. Although it is not possible to write contracts contingent on individual actions or the team performance, the principal can commit to a contract specifying nonnegative payments to each agent contingent on each possible project outcome. The principal's goal is to design this contract in a way that maximizes profit: revenue minus compensation.

The setting builds on the classic Holmström (1979) model, in which a single agent produces work of a non-contractible quality resulting in an observable outcome.[1] In our setting, the analogue of quality is a jointly achieved performance. How incentive spillovers across agents matter for contract design—a central issue for modern firms—is not well understood, despite the immense amount learned about contract design since Holmström's work. In this paper, we make progress on this problem by leveraging some ideas from network theory.

To illustrate the basic importance of incentive spillovers, imagine that the principal slightly adjusts the contract of a particular agent, Bob, in a way that motivates him to work harder. In team production, changing one team member's action can change other agents' private returns to effort, holding fixed their own contracts. Those whose efforts

[1] In particular, the obstacles to perfect contracting are the same: moral hazard and limited liability.

are complements to Bob's are now motivated to work harder, while those whose actions are substitutes have incentives to free-ride on his higher effort. This interconnectedness plays a pivotal role in the optimal allocation of incentive pay. In view of what we have just said, optimal contracts cannot be based only on the isolated contributions of agents' efforts to team performance. Contract design must also take into account agents' organizational positions—how their effort shapes other agents' responses to their own contracts.

Our contribution is a characterization of optimal contracts in the presence of incentive spillovers. The characterization is stated in terms of three kinds of quantities that can be associated to each agent at any contract. The first quantity, an agent's *marginal productivity*, is the partial derivative of team performance in an individual's action, holding others' actions fixed. The second quantity is called an agent's *centrality*: a measure of connectedness[2] in a network reflecting incentive spillovers, with connectedness to more productive agents weighted more. The relevance of these quantities for contract optimality should be intuitive in view of what we have said. The third quantity is an agent's *marginal utility of money*: it accounts for the fact that an agent who has a low valuation of an additional dollar is, all else equal, a less responsive and less appealing recipient of incentive pay.

Our main result is that, when the binding incentive constraints are local, optimal contracts satisfy a balance condition: the product of the three quantities described above is equal across all agents receiving any incentive pay. It turns out that the balance condition is necessary if the principal does not want to shift compensation across agents in any state—something that must hold at any optimum. The condition is interpretable, identifying the quantities that must be measured to assess the optimality of a contract. If the optimality condition is not satisfied, our results yield guidance for modifying a suboptimal incentive scheme to a better one. Indeed, a key underlying technical result computes the marginal benefit to the principal of increasing incentive pay to an agent in any contract (optimal or not). This marginal benefit turns out to be proportional to the product of the three terms.

[2]Operationalized as a weighted walk count, i.e., a Katz–Bonacich centrality.

It is worth emphasizing that our general model makes no parametric assumptions and thus allows quite flexible production functions for the team. To take just one example, the team's production function could be an arbitrary polynomial, with monomials of arbitrary degree reflecting outputs generated through the joint efforts of arbitrary subsets of the team—e.g., complementing one another in threes, fours, and so on. Nevertheless, the team's production function matters only through its first derivatives and its Hessian at the optimal contract—which measures bilateral spillovers (complementarities/substitutabilities) between agents. This allows us to leverage some methods from well-understood network games whose payoffs are quadratic polynomials to analyze the spillover effects of locally perturbing incentive pay under arbitrary contracts. We use this key reduction to characterize the first-order conditions that determine the principal's optimal allocation of incentives both across agents and across outcomes.

The first half of the paper presents the general result on the balance condition and explains the reasoning behind it. The second half of the paper explores a variety of implications when we specialize the model. We start with an intuitive one on ranking agents' incentive pay. Generally, agents with a higher product of productivity and centrality must have a lower marginal utility of money. If all agents have the same utility functions of money and these functions are concave, then those with the higher "productivity times centrality" index must be paid more in every outcome—a simple but novel rule of thumb.

The three factors in the balance characterization depend on the contract, so a natural next step is to give explicit characterizations of optimal contracts in terms of primitives only. We do so when team performance is given by some standard production functions. Two such cases are Cobb–Douglas and constant elasticity of substitution production functions; these functional forms have not yielded tractable analyses in the literature on network games and spillovers, but it turns out that they admit nice solutions for our contract-optimization problems. We show that when team members' efforts are more substitutable, the optimal contract is more unequal, focusing incentives more on more productive agents; when the efforts are more complementary, the contract spreads payments out across agents.

Optimal contracts can respond to the environment in counterintuitive ways. We demonstrate this in a third parametric application—a simple environment where the team's performance is given by a quadratic production function, yielding a canonical quadratic network game among the agents (Ballester, Calvó Armengol, and Zenou, 2006) for any fixed contract. In this special case of the model, agents have linear utilities of money and the output is equal to the sum of individual efforts plus a quadratic polynomial that we can identify with an exogenous network of bilateral productive complementarities. We show that the optimal contract *equalizes* all agents' spillovers on their neighbors in a suitable sense, which entails muting the incentives of more technologically central agents, all else equal. The contracts that achieve this can be computed explicitly and turn out to be quite different from those that are optimal in related models—e.g., Claveria-Mayol, Milán, and Oviedo-Dávila (2024)—where incentives are allocated in proportion to Katz–Bonacich centralities in an exogenous network. Our balance condition thus turns out to have some implications challenging standard intuitions in this setting.

The canonical quadratic game setting also facilitates some comments on the extensive margin of our problem—the "team design" question of which agents should be given incentive pay at all. For a given network of technological complementarities, we find that the principal may optimally give steep incentives to tightly knit teams with strong internal complementarities and exclude many others from incentive pay entirely.

Finally, we study a setting where contracts are constrained to take specific forms, namely "equity" contracts that give an agent the same share of the output in every state. We show that with this restriction, a version of our balance characterization holds, with corresponding rankings of incentive pay across agents. This illustrates that our analysis of incentive spillovers and the optimization of across-agent allocation is flexible about how optimization is handled across states. In particular, it does not rely on full contract flexibility and can accommodate some realistic constraints.

Related literature. In the contract theory literature, the Holmström (1979) model—studying incentives for a single agent under moral hazard and imperfect observability—is a special case of our multi-agent setup. We use the first-order approach (see Rogerson

(1985) and Jewitt (1988)). To our knowledge, there is not much work on how first-order conditions for contract optimality depend on spillovers.[3] Indeed, the extensive literature on moral hazard beginning with Holmström (1982) focuses mainly on different questions. In that strand, a key question is how a principal can use observed outcomes to separately detect agents' deviations from a desired action profile, often a nearly first-best one (see, e.g., Mookherjee, 1984; Legros and Matsushima, 1991; Legros and Matthews, 1993). Several features of our model prevent such schemes.[4] In this type of situation, when observability and fine-grained auditing of effort are fundamentally constrained, we examine how optimal contracts depend on spillovers in the production function.

Some of the closest work on optimal incentives in the presence of spillovers is found in the literature on networks. This includes, in addition to work already mentioned, papers such as Candogan, Bimpikis, and Ozdaglar (2012), Bloch (2016), Belhaj and Deroïan (2018), Galeotti, Golub, and Goyal (2020), Gaitonde, Kleinberg, and Tardos (2020), and Shi (2022). Our main contribution to this literature is a study of a natural non-parametric formulation, both in terms of the production function and the form of incentives. We show that network game techniques permit some general characterizations of optimal contracts without the parametric assumptions common in the network games literature. When we specialize to a canonical parametric environment in Section 5, we also contrast the more specific implications of our analysis with existing parametric networks models—of which the closest is the contemporaneous work by Claveria-Mayol, Milán, and Oviedo-Dávila (2024) on optimal linear incentive contracts in quadratic network games. Both our analysis and results end up being quite different.

The problem of designing multi-agent contracts has also recently attracted attention in a new algorithmic contract theory literature—e.g., Dütting, Ezra, Feldman, and Kesselheim (2023), Ezra, Feldman, and Schlesinger (2024), and Dütting, Ezra, Feldman, and Kesselheim (2025). A starting point of this work is that many standard approaches to finding optimal team contracts may make heavy demands on the analyst's knowledge of the entire production function and ability to perform computations on it. This literature studies environments with finitely many actions where combinatorial problems

[3]Itoh (1991) allows for a form of spillovers in a two-agent model.

[4]In particular, the contractible outcome (which has only finitely many possible values) is stochastically determined by a one-dimensional team performance, and there is limited liability.

create obstacles to tractable optimization and examines whether contracts can be devised that achieve some fraction of optimal performance. Our approach is very different methodologically in that actions and team performance are continuous, but the results give a perspective—complementary to the algorithmic contract theory work—on parsimonious ways to assess and improve on contract performance. Recent work by Zuo (2024), which discusses the structure of optimization problems in a model closely related to some of our special cases, shows that there are interesting computational questions in our more continuous type of setting as well.

2. Model

We first present the formal structure of the model and then, in Section 2.2, we comment on some issues of interpretation.

There are n agents, $N = \{1, 2, \ldots, n\}$. The agents take real-valued actions $a_i \geq 0$, which can be interpreted as effort levels. These jointly determine a team *performance*, given by a function $Y : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$, which we assume is twice differentiable and strictly increasing in each of its arguments. This team performance determines the project *outcome*, an element of the finite set $\mathcal{S}$. The probability of the outcome s is $P_s(Y)$, where for any $s \in \mathcal{S}$, the function $P_s(\cdot)$ is strictly positive and twice differentiable.[5]

There is also one principal. (When we use pronouns, we use "she" for the principal and "he" for an agent.) The principal receives *revenue* v_s from the outcome s.[6] The principal can make payments contingent on the project outcome but not on agents' efforts or the team performance Y. The principal commits to a non-negative payment contingent on the outcome. If outcome s is realized, the principal pays $\tau_i(s)$ to agent i. The function $\boldsymbol{\tau} : \mathcal{S} \to \mathbb{R}^n_{\geq 0}$ is called a *contract*.

We consider risk-averse agents and a risk-neutral principal.[7] The utility to agent i from a monetary transfer is given by the function $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is strictly

[5] The assumption that a probability of outcome function is strictly positive is not crucial to the results. It is only imposed to simplify some statements. In the absence of this assumption, our results would hold at outcomes that occur with non-zero probability at the optimal team performance.

[6] This should be interpreted as the principal's valuation of that outcome realizing, gross of any payments she will make to the agents.

[7] The modeling assumption that a principal is risk-neutral is not crucial to the results. The characterization of an optimal contract and its consequences can be straightforwardly extended to the case of a risk-averse principal.

increasing, concave, and differentiable. Each agent also has a cost-of-effort function $C_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is strictly increasing, strictly convex, and twice differentiable in that agent's action. The marginal cost at zero action is zero: $C_i'(0) = 0$. Agent i chooses a_i to maximize his expected payoff from payments minus his cost,

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} P_s\left(Y(a_i, a_{-i})\right) u_i\left(\tau_i(s)\right) - C_i(a_i).$$

The payoff for the principal given a contract $\boldsymbol{\tau}$ and team performance Y is the expected payoff of the outcome minus transfers to agents:

$$\sum_{s \in \mathcal{S}} \left(v_s - \sum_i \tau_i(s)\right) P_s(Y).$$

The timing is as follows: The principal commits to a contract $\boldsymbol{\tau}$, and then agents simultaneously choose actions. Our solution concept for the game among the agents is pure strategy Nash equilibrium; in the remainder of the paper, we simply use the word *equilibrium* to refer to this solution.

There may be multiple equilibria under some contracts. Given a contract $\boldsymbol{\tau}$, we assume that agents play an equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau})$ maximizing the principal's expected payoff. Under this selection, a principal's payoff under a contract is well-defined if at least one equilibrium exists. Among such contracts, a contract $\boldsymbol{\tau}$ is *optimal* if no other contract $\widetilde{\boldsymbol{\tau}}$ gives the principal a higher payoff. Implicit in this definition is the assumption that contracts without equilibria can never be optimal.

Our analysis will not rely on the existence of an optimal contract, but the following argument shows an optimal contract exists if we impose a bit of additional structure.

Fact 1. Suppose that the space of contracts giving the principal a non-negative payoff is compact.[8] Then an optimal contract exists.

As examples, this holds if the outcome is binary or if the outcome probabilities $P_s(Y)$ are uniformly bounded away from zero. The proof uses a compactness argument, taking a sequence of contracts whose payoffs converge to the supremum of attainable principal payoffs, along with their corresponding equilibria. By compactness of the contract space and action space, we can extract convergent subsequences of both the contracts

[8]We exclude any contracts where no equilibrium exists from this space.

and equilibrium actions. The limit contract achieves the supremum payoff because equilibria are upper-hemicontinuous in the contract and the principal's payoff function is continuous in both contracts and actions.

2.1. **Simple success-or-failure environments.** We introduce a class of *simple success-or-failure environments* that will be helpful for illustrating our results.

There are two possible outcomes $s \in \{0, 1\}$. The revenues from these outcomes are normalized so that $v_1 = 1$ and $v_0 = 0$. These can be interpreted as success or failure of the project. The probability of success is $P(Y)$, where the function $P(\cdot)$ is strictly increasing and twice differentiable.

Each agent has a quadratic cost of effort.[9] Agent i maximizes the expected payoff given by the expression:

$$\mathcal{U}_i = P(Y)\tau_i(1) + (1 - P(Y))\,\tau_i(0) - \frac{a_i^2}{2}.$$

Fact 2. In simple success-or-failure environments, it is optimal for the principal to pay nothing at the failure outcome—that is, $\tau_i(0) = 0$ for all agents i. A contract can then be represented by a n-dimensional vector $\boldsymbol{\tau} \in \mathbb{R}^n_{\geq 0}$ consisting of payments in the success outcome.

The reason is simple: agents' incentives depend only on the difference $\tau_i(1) - \tau_i(0)$ between transfers conditional on success and failure, so we can shift payments and assume $\tau_i(0) = 0$. This shift can only improve the principal's payoff, so it is without loss of optimality in the principal's problem. It is without loss of generality to assume the value v_1 of success is 1, and we can then interpret $\tau_i(1)$ as an equity share in the project's output.

Within this class of environments, we can define an important leading example, which we will analyze fully in Section 5.

Example 1. Fix a symmetric matrix $\boldsymbol{G}$ representing an undirected network, with $G_{ij} \geq 0$ being the weight of the link from agent i to j, and $G_{ii} = 0$ for each i. The team performance is the sum of a term that is linear in actions—corresponding to agents'

[9]This is not substantively restrictive in that, under smoothness assumptions, a monotone transformation can be applied to a_i to achieve this form.

standalone contributions—and a quadratic complementarity term:

$$Y(\boldsymbol{a}) = \sum_{i \in N} a_i + \frac{1}{2} \sum_{i,j \in N} G_{ij} a_i a_j. \tag{1}$$

The structure of this example is simple in several respects: team performance is quadratic, the outcome is binary, and all agents' efforts are complementary.

2.2. **Remarks on the model.** The team performance Y is real-valued, but the outcome s is discrete. This assumption need not substantively restrict the scope of the model since the outcome can be, for example, a revenue rounded to the nearest cent. On the other hand, the fact that outcomes are mediated by a one-dimensional performance level is important, though, as we discuss in the paper's concluding remarks, the key ideas have implications for outcomes determined by a higher-dimensional function of efforts.

We assume that the firm's output is the only contractible consequence of any agent's effort. In other words, agents cannot be paid directly for their efforts a_i. In this we follow the literature stemming from Holmström (1982), which is motivated by the practical features of contracts and the fact that a_i and Y are abstractions that may not have any specific (and certainly not any feasibly measurable) real-world counterpart.

The principal is not restricted to a budget of v_s at outcome s. The principal may be willing to lose money at some outcomes with the hope of inducing a higher action.

We do not assume that equilibria exist under all contracts in the formulation of the model or our analysis. For the contract that pays zero in all states, there is always an equilibrium in which agents take zero actions; this ensures that the principal's value is well-defined. For other contracts, existence needs to be analyzed in the environment of interest; for example, we show equilibria exist for all contracts in the parametric model in Section 5.

3. Optimal Contracts

This section states our main result characterizing optimal contracts. Defining these quantities in general is a bit involved, and so to motivate the definitions, we now preview them in the special case of simple success-or-failure environments.

Consider any contract; let $\boldsymbol{\tau}$ be the vector of payments in case of success (recalling that they are 0 in case of failure). Also, fix an associated equilibrium $\boldsymbol{a}^*$. Define

$$\alpha_i = \frac{\partial Y}{\partial a_i} \quad \text{and} \quad G_{ij} = \frac{\partial^2 Y}{\partial a_i a_j}$$

to be, respectively the marginal productivity of i at $\boldsymbol{a}^*$ and the spillover of i's effort onto j's productivity. Write Y for $Y(\boldsymbol{a}^*)$, define the diagonal matrix $\boldsymbol{T} = \mathrm{diag}(\boldsymbol{\tau})$ (with diagonal entries T_{ii}) and let

$$\boldsymbol{c}^\top := \boldsymbol{\alpha}^\top \left[\boldsymbol{I} - P'(Y)\boldsymbol{T}\boldsymbol{G}\right]^{-1}.$$

This is a standard network centrality measure—the Katz–Bonacich centrality vector in network $\boldsymbol{TG}$ with weight vector $\boldsymbol{\alpha}$, with decay factor $P'(Y)$. Agent i's centrality c_i can be interpreted as a measure of connectedness—operationalized as a weighted walk count—in the network $\boldsymbol{TG}$, with connectedness to more productive agents weighted more (see Ballester et al. (2006) and Bloch, Jackson, and Tebaldi (2023)).

A key point of our analysis is that for local perturbations of incentives, the movement of the equilibrium yields a change in performance satisfying

$$\frac{dY}{d\tau_i} \;\propto\; \alpha_i \cdot c_i \;=\; \text{productivity}_i \cdot \text{centrality}_i.$$

That is, increasing i's incentive pay increases output proportional to i's productivity times i's centrality. Therefore, at an optimal contract, these products must be equal across all agents receiving positive incentive pay. This basic "productivity times centrality" formula will appear in the principal's first-order conditions more generally, along with a third factor that will arise when we allow utilities that are not linear in money.

We now return to the general setting and give our main definitions.

3.1. **Key objects.** Fix a contract $\boldsymbol{\tau}$ and a corresponding principal-optimal equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau})$. Let Y^* be the team performance under this equilibrium. We define a series of objects below—at this equilibrium, under this contract—but in many cases we omit the dependence on the equilibrium and the contract for brevity.

We will define general versions of the productivity vector $\boldsymbol{\alpha}$ and the spillover matrix $\boldsymbol{G}$ that appeared in our motivating formulas above. There, costs had a special property: the cost of effort satisfies $C_i''(a_i) = 1$. We introduce a normalization to mimic this in a general environment. Let the *curvature matrix* $\boldsymbol{H}$ be a diagonal matrix where

$$H_{jj} := C_j''(a_j^*)$$

is the second derivative of the cost function for agent j at $\boldsymbol{a}^*$. In general, when we analyze how equilibrium actions vary with contract perturbations, an agent's best response is less sensitive to increased incentives when $C_j''(a_j^*)$ is larger, and $\boldsymbol{H}$ will help us capture this effect.

Let $\nabla Y(\boldsymbol{a}^*)$ be the gradient of $Y(\cdot)$ at $\boldsymbol{a}^*$, restricted to the agents that take a strictly positive action. We define the *(normalized) marginal productivity* vector $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha} := \boldsymbol{H}^{-\frac{1}{2}} \nabla Y(\boldsymbol{a}^*).$$

The i^{th} element α_i captures the marginal effect of i's action on team performance, rescaled to adjust for the curvature of i's cost function.

To analyze how incentives propagate through the team, we consider the Hessian matrix of the team performance function $Y(\cdot)$ with respect to agent actions. Let $\boldsymbol{G}$ denote the Hessian matrix of Y restricted to agents that take a strictly positive action in $\boldsymbol{a}^*$. Formally, for agents j and k such that $a_j^* > 0$ and $a_k^* > 0$, define

$$G_{jk} := \frac{\partial^2 Y}{\partial a_k \partial a_j}.$$

Let the *marginal payment utility* matrix $\boldsymbol{U}$ be a diagonal matrix where

$$U_{jj} := \sum_{s \in \mathcal{S}} P_s'(Y^*) u_j(\tau_j(s))$$

is the marginal change in agent j's utility from payments when team performance increases. The increase in Y changes all the probabilities of outcomes, and the agent's utility from these outcomes is given by $u_j(\tau_j(s))$, where the contract is held fixed.

We next define an object $\boldsymbol{c}$ that will capture how a change in an agent's incentives propagates through the team:

$$\boldsymbol{c}^T := \boldsymbol{\alpha}^T \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}} \boldsymbol{U} \boldsymbol{G} \boldsymbol{H}^{-\frac{1}{2}} \right]^{-1}. \tag{2}$$

The i^{th} element c_i of this vector is the total effect on team performance induced by a marginal change in agent i's incentive to increase a_i. This effect is inclusive of all spillovers on others' incentives through strategic interactions.

In simple success-or-failure environments, the formulas recover the simple expressions that we started the section with. In that case, the matrix $\boldsymbol{H}$ is equal to the identity matrix $\boldsymbol{I}$ because the cost of each agent is $\frac{1}{2}a_i^2$. If we further specialize to the setting of Example 1, then the Hessian matrix $\boldsymbol{G}$ is equal to the matrix in the production function

$$Y(\boldsymbol{a}) = \sum_{i \in N} a_i + \frac{1}{2} \sum_{i,j \in N} G_{ij} a_i a_j,$$

and productivities are given by $\boldsymbol{\alpha} = \mathbf{1} + \boldsymbol{G}\boldsymbol{a}$. The factors involving the matrix $\boldsymbol{H}$ account for the fact that a higher curvature of costs attenuates strategic responses and thus spillovers.

3.2. **Balance condition across agents.** In this section, we present our main result: a balance condition across agents at each outcome realization. Our analysis will characterize optimal contracts under the following assumption.

Assumption 1. *A differentiable selection $\boldsymbol{a}^*(\boldsymbol{\tau})$ from the equilibrium correspondence can be defined in a neighborhood of $\boldsymbol{\tau}^*$.*

This assumption stipulates that equilibrium varies differentiably as we slightly perturb the contract in a neighborhood of the optimum. We discuss when the assumption holds and what a first-order approach can tell us without the assumption in the following subsection. Our main result is a necessary condition characterizing optimal contracts, subject to Assumption 1:

Theorem 1. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and Y^* is the induced team performance. There exist constants λ_s such that for any agent i receiving a positive payment under an outcome s, we have*

$$\alpha_i c_i u_i'(\tau_i^*(s)) = \lambda_s.$$

Moreover, the outcome-dependent constants λ_s satisfy $\lambda_s \propto \frac{P_s(Y^*)}{P_s'(Y^*)}$.

This result says that optimal incentives require balance to hold, with the product on the left being equal across agents. More informally, the balance condition states that under an outcome s,

$$\text{productivity}_i \cdot \text{centrality}_i \cdot \text{marginal utility}_i \;=\; \text{constant}$$

for all agents paid when that outcome occurs. Below, we will give more intuition for why this is a necessary condition.

In fact, the proof does not rely on the induced team performance Y^* being optimal. The balance condition at the optimal contract would hold if the principal instead wanted to implement any desired level of performance with minimal (expected) transfers to agents.

The key to the proof, formalized in the following lemma, is calculating the effect on team performance of increasing an agent's payment under a given outcome. Assumption 1 ensures that these perturbations are well-defined.

Lemma 1. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract with corresponding equilibrium actions $\boldsymbol{a}^*$ and team performance Y^*. Consider any agent i receiving a positive payment at some outcome. For any outcome s, the derivative of team performance in $\tau_i(s)$, evaluated at $\boldsymbol{\tau}^*$, is*

$$\frac{dY}{d\tau_i(s)} = lP_s'(Y^*)\alpha_i c_i u_i'(\tau_i^*(s)),$$

where l is independent of i and s.

A complete proof for the result above is provided in Appendix A. We provide some intuition for the various terms in the expression in the lemma.

Intuition for the proof: The lemma characterizes the effect on team performance of increasing the transfer to agent i under outcome s. We can decompose this effect as the product of:

(i) a factor $P_s'(Y^*)\alpha_i$ capturing the sensitivity of the probability of the outcome s to i's effort;
(ii) a factor $u_i'(\tau_i(s))$ capturing the direct effect of increasing $\tau_i(s)$ on i's utility at outcome s;
(iii) a term c_i capturing the spillovers from changing i's incentive to exert effort;
(iv) the constant l, which depends on the curvature of the probability $P_s(Y)$.

We focus on the first three factors and defer treatment of the fourth term, which is not central in the basic intuition, to the formal proof in the appendix.

The first two factors measure the principal's ability to directly incentivize agent i by rewarding that agent when outcome s is realized. The change in agent i's marginal utility of action as $\tau_i(s)$ increases slightly is the product of the marginal effect $P_s'(Y^*)$ of team performance on the probability of outcome s, the effect $\frac{\partial Y}{\partial a_i}$ of i's action on team performance, and the marginal utility $u_i'(\tau_i(s))$ of money under outcome s. To obtain agent i's direct response to a stronger incentive, we divide by the curvature $C_j''(a_j^*)$ of the cost (recall $C_j''(a_j^*)$ is the denominator of α_i); the curvature of the agent's cost plays a role for the same reason that the curvature of his utility of money does.

Multiplying by the second term translates from this direct effect on i's action to the overall change in equilibrium actions. The term c_i measures the total spillovers induced by shifting i's incentive to exert effort. Recall the definition

$$\boldsymbol{c}^\top := \boldsymbol{\alpha}^\top \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1}.$$

When $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ has spectral radius less than one, the expansion

$$\left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1} = \sum_{k=0}^{\infty}(\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}})^k,$$

gives a helpful intuition. The powers capture the initial increase in i's action, the resulting changes in each agent's best response, the further changes in best responses induced by these, etc. Thus the full summation captures the change in the equilibrium action profile due to the exogenous change in i's incentives—following a standard intuition in network games (Ballester et al., 2006). Finally, the dot product with the marginal productivity vector $\boldsymbol{\alpha}$ translates this change in actions into the change in team performance.

We next discuss some intuition for why Lemma 1 implies Theorem 1. A formal proof is provided in Appendix A. We want to show that the balance condition

$$\alpha_i c_i u_i'(\tau_i(s)) = \alpha_j c_j u_j'(\tau_j(s)),$$

must hold under an optimal contract. Suppose that the principal would benefit from a slightly higher team performance (the case in which the principal prefers a slightly lower team performance proceeds analogously). Lemma 1 shows that the change in team performance from increasing agent i's payment under outcome s is equal to $\alpha_i c_i u_i'(\tau_i(s))$ times terms independent of i, and similarly for agent j. If we had

$$\alpha_i c_i u_i'(\tau_i(s)) > \alpha_j c_j u_j'(\tau_j(s)),$$

it would be profitable for the principal to pay agent i slightly more and agent j slightly less under outcome s. The same argument holds in the opposite direction, so the balance condition is necessary for the contract to be optimal.

As initial illustrations of the theorem, we discuss its consequences in examples with Cobb–Douglas and constant elasticity of substitution production.

Example 2 (Cobb–Douglas). We work again in the simple success-or-failure environment. Suppose all agents are risk-neutral and team performance is

$$Y(\boldsymbol{a}) = \prod_{i=1}^{n} a_i^{\gamma_i},$$

where the exponents γ_i, which we will call the factor shares, can differ across agents. We assume all agents have the same quadratic effort cost $C_i(a_i) = a_i^2/2$, but it is straightforward to extend the subsequent analysis to heterogeneous quadratic costs by rescaling actions. One can directly characterize the optimal contract by applying Theorem 1 (see Zuo (2024), which solves a closely related example). We take an alternate approach of transforming our problem to a simpler one and then applying Theorem 1.

Consider an equivalent transformed problem in which we replace Y with $\widetilde{Y} = \log(Y)$ and $P(\widetilde{Y})$ with $\widetilde{P}(Y) = P(\exp(\widetilde{Y}))$. The problem is now separable:

$$\widetilde{Y}(\boldsymbol{a}) = \sum_{i=1}^{n} \gamma_i \log(a_i).$$

This transformation does not change the optimal contract or the corresponding equilibrium actions.

Recall that it is without loss to consider contracts described by payments τ_i to each agent conditional on success (and making no payments conditional on failure). Each agent's first-order condition is

$$a_i^* = \frac{\tau_i \widetilde{P}'(\widetilde{Y}^*)\gamma_i}{a_i^*} \tag{3}$$

We now compute the quantities in Theorem 1. Differentiating $\widetilde{Y}$, we compute marginal productivities to be

$$\widetilde{\alpha}_i = \frac{\gamma_i}{a_i^*}.$$

Since the transformed problem is separable, the matrix $\frac{\partial^2 \widetilde{Y}}{\partial a_i \partial a_j}$ of spillovers is diagonal with entries $\frac{\partial^2 \widetilde{Y}}{\partial a_i^2} = -\frac{\gamma_i}{a_i^2}$. We next compute agents' centralities at equilibrium to be

$$\begin{aligned} \widetilde{c}_i &= \widetilde{\alpha}_i \left(1 + \frac{\tau_i \widetilde{P}'(\widetilde{Y}^*)\gamma_i}{(a_i^*)^2}\right)^{-1} \\ &= \frac{\widetilde{\alpha}_i}{2} \text{ by eq. (3)}. \end{aligned}$$

We can now apply Theorem 1, which states that under any optimal contract satisfying Assumption 1, the quantity $\frac{\widetilde{\alpha}_i^2}{2}$ is equal for all agents and thus all agents have the same marginal productivity. This tells us that an agent's equilibrium action is proportional to γ_i, the agent's factor share in the original production function. Given this, Equation (3) implies that each agent's payment τ_i under the optimal contract is proportional to his factor share γ_i.

A similar approach allows us to characterize optimal contracts under constant elasticity of substitution production.

Example 3 (Constant elasticity of substitution)*.* The setup is the same as in the previous example except that production is now

$$Y(\boldsymbol{a}) = \left(\sum_{i=1}^{n} \gamma_i a_i^{\rho}\right)^{\kappa/\rho}$$

for a non-zero ρ. The parameter κ captures returns to scale and the coefficients γ_i, which we also call factor shares, can differ across agents. Finally, $1/(1-\rho)$ is the elasticity of substitution between agents' efforts. It is again useful to transform team production: we consider an equivalent transformed problem in which we replace Y with $\widetilde{Y} = \frac{1}{\rho} Y^{\rho/\kappa}$ and $P(Y)$ with $\widetilde{P}(\widetilde{Y}) = P((\rho \widetilde{Y})^{\kappa/\rho})$. The transformed problem is:

$$\widetilde{Y}(\boldsymbol{a}) = \frac{1}{\rho} \sum_{i=1}^{n} \gamma_i a_i^{\rho}. \tag{4}$$

This transformation again does not change the optimal contract or the corresponding equilibrium actions. The role of ρ in the transformation is to ensure that $\widetilde{Y}$ is an increasing function of actions.

We again can consider contracts described by payments τ_i to each agent conditional on success. A complication is that it need not be optimal to pay all agents positive transfers τ_i. For example, when $\rho \geq 1$, optimal contracts will only pay the agent(s) with the highest factor share(s) γ_i. We characterize how the optimal contract divides payments among agents who do receive positive payments. Each of these agents have first-order conditions

$$a_i^* = \tau_i \widetilde{P}'(\widetilde{Y}^*)\gamma_i \cdot (a_i^*)^{\rho-1}. \tag{5}$$

We next compute marginal productivities and centralities. Differentiating $\widetilde{Y}$, marginal productivities are

$$\widetilde{\alpha}_i = \gamma_i a_i^{\rho-1}.$$

The matrix $\frac{\partial^2 \widetilde{Y}}{\partial a_i \partial a_j}$ of spillovers is again diagonal with entries $\frac{\partial^2 \widetilde{Y}}{\partial a_i^2} = \gamma_i(\rho-1)a_i^{\rho-2}$. So agents' centralities at equilibrium are

$$\begin{aligned}\widetilde{c}_i &= \widetilde{\alpha}_i \left(1 - \tau_i \widetilde{P}'(\widetilde{Y}^*)\gamma_i(\rho-1)\cdot(a_i^*)^{\rho-2}\right)^{-1} \\ &= \frac{\widetilde{\alpha}_i}{2-\rho} \text{ by eq. (5).}\end{aligned}$$

We can now apply Theorem 1, which states that under any optimal contract satisfying Assumption 1, the quantity $\frac{\widetilde{\alpha}_i^2}{2-\rho}$ is equal for all agents and thus all agents have the same marginal productivity. This tells us that $(a_i^*)^{1-\rho}$ is proportional to γ_i. Given this, (5) implies that each agent's payment τ_i under the optimal contract is proportional to $\gamma_i^{\frac{1}{1-\rho}}$.

Recall that γ_i is the factor share of agent i and $\frac{1}{1-\rho}$ is the elasticity of substitution. So the principal pays more to agents with higher factor shares, and this effect is amplified when inputs are more substitutable and dampened when inputs are less substitutable. An intuition for this can be seen by considering two cases. In the limit as $\rho \to -\infty$, the production function converges to the Leontief production function $Y(\boldsymbol{a}) = \min(a_i)$. In this case, it is optimal to pay all agents equally because inducing one agent to take a higher action than others does not improve team performance. In the limit as $\rho \to 0$, the production function approaches Cobb–Douglas, where productivities in (4) are much more responsive to own effort, and contribute to transformed output in proportion to γ_i. Thus compensation is linear in γ_i (per the previous example).

We will see in Section 5 that Theorem 1 leads to an explicit characterization of contracts when production is determined by a standard network game; the previous two examples show it also gives explicit characterizations for production functions that have been difficult to analyze in the network games literature. A key point is that because we assume outcomes in our model depend on a one-dimensional team performance (via functions $P_s(Y)$ that can be quite general), we can apply monotone transformations that simplify the relevant spillovers between agents.

3.3. **Differentiability of equilibrium.** Before turning to consequences of Theorem 1, we briefly discuss the substantive meaning of Assumption 1 and what can be said under weaker assumptions. The assumption is implied by the following two conditions:

Assumption 2. *(a) (Invertibility of utility Hessian) The matrix*

$$\left(\frac{\partial^2 \mathcal{U}_i}{\partial a_j \partial a_i}\right)_{i,j}$$

is non-singular at contract $\boldsymbol{\tau}^*$ and corresponding equilibrium $\boldsymbol{a}^*$.
(b) (Strictness) The equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$ is strict.

This assumption gives more explicit conditions that guarantee a first-order approach applies. Part (a) is weaker than requiring stability of equilibrium under best-reply dynamics. Part (b) can impose more substantive restrictions, as we explain below. We first show Assumption 2 implies Assumption 1 and then discuss relaxing part (b).

When part (a) of Assumption 2 holds, the implicit function theorem lets us define action profiles $\boldsymbol{a}(\boldsymbol{\tau})$ in a neighborhood of $\boldsymbol{\tau}^*$ such that all agents' first-order conditions are satisfied and $\boldsymbol{a}(\boldsymbol{\tau})$ is continuously differentiable in $\boldsymbol{\tau}$. When the equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau})$ is strict (i.e., part (b) of Assumption 2 also holds), the solutions to the first-order conditions $\boldsymbol{a}(\boldsymbol{\tau})$ must be equilibria in a neighborhood of $\boldsymbol{\tau}^*$.

The equilibrium $\boldsymbol{a}^*$ under $\boldsymbol{\tau}^*$ is strict when all agents have a unique best response, as in Section 5 as well as other environments where the costs of effort are sufficiently convex. But environments that do allow indifferences can pose obstacles. If one or more agents are indifferent to their equilibrium action and an alternate action that may be far away, then perturbing $\boldsymbol{\tau}^*$ in some directions could make an alternate action more desirable. So the solution $\boldsymbol{a}(\boldsymbol{\tau})$ to agents' first-order conditions need not be optimal actions for all agents. In this case, the balance conditions in Theorem 1 may not hold; our analysis there only holds when the local incentive constraints are the binding ones.

Nevertheless, as we show in Appendix B, if contract perturbations in *some* directions maintain the existence of an equilibrium near the one of interest,[10] we can obtain modified balance conditions in those directions. The intuition is that the principal must be indifferent to perturbations to $\boldsymbol{\tau}^*$ that keep the agent indifferent between his equilibrium action and the best alternate action. So a first-order condition approach to characterizing optimal contracts can be more broadly relevant.

4. Comparisons across agents and outcomes

This section derives consequences of the main result for a comparison of payments made across agents and across outcomes. Section 4.1 shows that agents with symmetric utility functions can be ranked in terms of payments at the optimal contract. Section 4.2 compares the payments a particular agent receives across different outcomes.

4.1. **Ranking agents at the optimal contract.**

Agents can be ranked in terms of payments at the optimal contract. To see this, we establish a relationship between the marginal utilities of agents. An implication of Theorem 1 is that the ratio between any two agents' marginal utilities is the same at every outcome such that both receive positive transfers.

Corollary 1. *Consider an optimal contract $\boldsymbol{\tau}^*$. Let $\mathcal{S}^*_{ij}$ be the set of outcomes at which agents i and j both receive a positive payment. For any outcome $s \in \mathcal{S}^*_{ij}$, we have*

$$\frac{u_i'(\tau_i^*(s))}{u_j'(\tau_j^*(s))} = \frac{\alpha_j c_j}{\alpha_i c_i}.$$

[10]More formally, if the equilibrium correspondence is differentiable when perturbations are confined to a nontrivial subspace.

Intuitively, since outcome probabilities are determined by a joint team performance, agents' incentives should vary across outcomes in similar ways. The corollary formalizes this intuition in terms of marginal utilities in each outcome.[11]

The corollary only applies when the set of outcomes at which agents i and j both receive a payment is non-empty. Determining when an agent is paid at a given outcome can be complicated in general, but it is easy to construct settings where the corollary applies. In Appendix C, for example, we give a class of environments in which an Inada condition guarantees that all agents are paid at all outcomes where $P_s'(Y^*) > 0$ (and no other outcomes).

When agents have identical utility functions, agents can be ranked so that an optimal contract provides stronger incentives to more highly ranked agents.

Proposition 1. *Suppose that $\boldsymbol{\tau}^*$ is an optimal contract. If a pair of agents i and j have identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$, then*

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

The intuition is simple: for two agents that derive the same value from a monetary transfer, the agent with a greater overall effect on team performance at the optimal contract must be receiving a higher payment.

When all agents have an identical utility function, the optimal contract induces a complete ranking on the agents. The relative magnitude of payments across agents depends on the environment. This becomes evident in the parametric example discussed further in Section 5.

4.2. **Payments across outcomes.**

A second implication of the main balance result is a relationship between a single agent's marginal utility across outcomes.

[11] This contrasts with a literature on optimal compensation when the observed outcome can be used to identify individuals who deviated from a desired level of effort (e.g., Holmström (1982) and Legros and Matthews (1993)).

Corollary 2. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and Y^* is the induced team performance. If agent i receives positive payments under outcomes s_1 and s_2, then*

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

The corollary states that the marginal utility under each outcome is proportional to the probability of that outcome divided by the marginal change in that probability as team performance increases. That is, agents are paid more in outcomes that are less likely and more responsive to team performance. This result generalizes a result in the single-agent setting of Holmström (1979) concerning how a (single agent's) payments should be allocated across outcomes.

A straightforward application of Corollary 2 characterizes the set of outcomes at which an agent receives a positive payment. If an agent receives a positive payment at some outcome, the outcomes at which it receives a positive payment must all either have a positive marginal probability at equilibrium team performance, or a negative marginal probability. When the team performance function $Y(\cdot)$ is strictly increasing in each of its arguments, the outcomes at which an agent receives a positive payment all have a positive marginal probability at equilibrium team performance. (This is formalized as Lemma 4 in the Appendix).

In the special case that an agent is risk-neutral, a stronger conclusion can be derived on the outcomes at which the agent is paid. Under a mild assumption on the probability of outcome function $P_s(\cdot)$, a risk-neutral agent receives a positive payment in at most one outcome.

Proposition 2. *Suppose that for an optimal contract $\boldsymbol{\tau}^*$ and induced team performance Y^*, there does not exist a pair of outcomes s_1 and s_2 such that*

$$\frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} = \frac{P_{s_2}(Y^*)}{P_{s_2}'(Y^*)}.$$

Then, any risk-neutral agent receives a positive payment in at most one outcome. Moreover, this outcome is unique across all risk-neutral agents.

Risk-averse agents prefer to diversify their payments across outcomes. But a risk-neutral agent does not have this diversification motive, and therefore is best motivated by payment in the outcome that responds most to the team's performance. When all agents are risk-neutral, the optimal contract makes a positive payment to the team at only one outcome. The condition on the functions $P_s(Y)$ holds at the endogenous team performance, but it is straightforward to construct functions $P_s(Y)$ such that the condition does not hold for any possible team performance Y.

5. Application to a canonical network game

Our main result gives necessary conditions for a contract to be optimal. This result, however, does not directly characterize how agents' incentives and equilibrium actions vary with the environment. Concretely, as we vary parameters affecting an agent's productivity or complementarities, the balance condition must remain satisfied at optimal contracts, but this might happen via adjustments in any of the three factors in the balance condition.

In this section, we perform a few exercises to better understand how balance affects observable outcomes of interest. First, we study a case where the adjustments can be fully and explicitly characterized—Example 1. The characterization is very different from classic results on network games with fixed incentives (Ballester et al., 2006). With fixed incentives, making an agent more central in the network of complementarities increases that agent's productivity. Optimal incentive provision provides a remarkable countervailing force that mutes technological asymmetries. Specifically, in that example, the balance condition is achieved by equalizing agents' endogenous marginal productivities and centralities, despite their different technological roles. We highlight some other notable implications: conflicts of interest that arise between the principal and the agents over technological improvements, which are again caused by optimally designed incentives for effort.

We then generalize the model of Example 1 to allow for different standalone productivities, which makes it possible for the model to produce across-agent productivity differences in equilibrium. We study comparative statics as standalone productivities are varied and show that contract optimality may require an inverse relationship between productivity and centrality, in the sense that making an agent more productive may force that agent to become central at the new equilibrium. This again illustrates some non-obvious testable implications of the model in particular instances of our setting.

5.1. **Equilibrium characterization.** Throughout Section 5, we study the setup of Example 1 from Section 2. Recall there are two outcomes: 1 (success) and 0 (failure). The probability of success is given by $P(\cdot)$ which is strictly increasing and twice differentiable. Throughout this section, we will additionally maintain the assumption that $P(\cdot)$ is strictly concave.

Recall that, by Fact 2 in Section 2.1, we may reduce the contract design problem to choosing a single vector $\boldsymbol{\tau} \in \mathbb{R}^n$ of payments conditional on success.

Proposition 3. *Fixing $\boldsymbol{\tau}$, there exists a unique Nash equilibrium. The equilibrium actions $\boldsymbol{a}^*$ and team performance Y^* solve the equations*

$$[\boldsymbol{I} - P'(Y^*)\boldsymbol{TG}]\boldsymbol{a}^* = P'(Y^*)\boldsymbol{\tau} \text{ and } Y^* = Y(\boldsymbol{a}^*), \tag{6}$$

where $\boldsymbol{T} = \operatorname{diag}(\boldsymbol{\tau})$ is the diagonal matrix with entries $T_{ii} = \tau_i$.

The characterization is reminiscent of the form of actions in standard network games, and extends the analysis of (Ballester et al., 2006) to a case with nonlinearities arising to P.[12] From this perspective, we can compare the analysis with existing work on planner interventions—for example Galeotti et al. (2020), or Parise and Ozdaglar (2023). Those papers typically posit a specific technology of intervention affecting the first (standalone) term in (1) rather than the complementarities term. Modeling monetary incentives in line with standard moral hazard theory yields an interestingly different problem, where planner interventions also affect the effective network of spillovers among the agents.

Note that the result entails a positive equilibrium action for those agents with $\tau_i > 0$, and an action of zero otherwise. An agent is said to be *active* under a given contract $\boldsymbol{\tau}$ if he receives a positive payment $\tau_i > 0$ and *inactive* otherwise. We will focus on characterizing the optimal allocation of shares among active agents. We discuss extensive margin questions in Section 6.

5.2. **Optimal contract.** We now characterize the optimal payments and equilibrium actions among the set of agents receiving positive shares.

[12]Indeed, dispensing with the interpretation of P as a probability, the same characterization works when the team members have shares in a common output $P(Y)$, with the $Y = Y(\boldsymbol{a})$ given by the formula of Example 1.

Proposition 4. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and $\boldsymbol{a}^*$ and Y^* are the induced equilibrium actions and team performance, respectively. The following properties are satisfied:*

(a) For any two active agents i and j, we have $\alpha_i = \alpha_j$ and $c_i = c_j$.
(b) Balanced neighborhood actions: There is a constant $\lambda' > 0$ such that for all active agents i, we have $(\boldsymbol{G}\boldsymbol{a}^*)_i = \lambda'$.
(c) Balanced neighborhood equity: There is a constant $\lambda > 0$ such that for all active agents i, we have $(\boldsymbol{G}\boldsymbol{\tau}^*)_i = \lambda$.

The result states that all active agents have equal marginal productivities and equal centralities.

The property of balanced neighborhood actions states that for each active agent i, the sum of actions of neighbors of i, weighted by the strength of i's connections to those neighbors in $\boldsymbol{G}$, is equal to the same number, λ'. Similarly, the property of balanced neighborhood equity says that for each active agent i, the sum $\sum_j G_{ij}\tau_j$ of shares given to neighbors of i, weighted by the strength of i's connections to those neighbors in $\boldsymbol{G}$, is equal to the same number (i.e., is not dependent on i).

Proof of Proposition 4. The characterization of optimal payments in Proposition 4 follows from the balance result derived in Theorem 1. To see this, first observe the following immediate corollary of Theorem 1 in the present environment, which follows from the theorem by observing $u'(\tau_i) = 1$ for all values of τ_i.

Corollary 3. *At an optimal contract $\boldsymbol{\tau}^*$, the product $\alpha_i c_i$ is a constant across all active agents.*

The following lemma is then the key step in proving Proposition 4.

Lemma 2. *If $\alpha_i c_i$ is constant across all active agents, then, α_i is constant across all active agents.*

The proof of this lemma, which we give in the appendix, starts by differentiating the production function and using the characterization of equilibrium, yielding the formula

$$\boldsymbol{\alpha} = \nabla Y(\boldsymbol{a}^*),$$

$$= \mathbf{1} + \boldsymbol{G}\boldsymbol{a}^*,$$
$$= \underbrace{[\boldsymbol{I} - P'(Y^*)\boldsymbol{G}\boldsymbol{T}]^{-1}}_{\boldsymbol{M}}\mathbf{1}.$$

Corollary 3 implies that the i maximizing α_i among active agents must minimize c_i. The fact $\boldsymbol{\alpha} = \boldsymbol{M}\mathbf{1}$, just derived, along with the definition (recall eq. 2) $\boldsymbol{c} = \boldsymbol{M}\boldsymbol{\alpha}$ can be combined to show that this is possible only if each entry of $\boldsymbol{\alpha}$ is equal. (In fact the proof of this fact uses only the two equalities just stated and that $\boldsymbol{M}$ is a nonnegative matrix.)

This conclusion implies part (b) of the proposition using the formula $\boldsymbol{\alpha} = \mathbf{1} + \boldsymbol{G}\boldsymbol{a}^*$ found above. To show (c), observe that the definition of $\boldsymbol{\alpha}$ and Lemma 2 imply there is λ_1 such that

$$(\mathbf{1}^T[\boldsymbol{I} - P'(Y^*)\boldsymbol{T}\boldsymbol{G}]^{-1})_i = \lambda_1$$

for all i. Therefore,

$$1 = \lambda_1 - P'(Y^*)\lambda_1(\boldsymbol{G}\boldsymbol{\tau})_i$$

for all i, so there exists a constant λ such that $(\boldsymbol{G}\boldsymbol{\tau})_i = \lambda$ for all i (among the subnetwork of active agents). $\square$

5.2.1. *An explicit characterization of the optimal contract.* The system of equations in part (a) of Proposition 4 can be solved explicitly for the optimal payments $\boldsymbol{\tau}^*$ as long as the relevant adjacency matrix $\boldsymbol{G}$ is invertible, which holds for generic weighted networks. At an optimal solution, the payment to an active agent i is

$$\tau_i^* \propto \left(\widetilde{\boldsymbol{G}}^{-1}\mathbf{1}\right)_i,$$

where $\widetilde{\boldsymbol{G}}$ is the subnetwork of active agents for that payment allocation; the same is true for actions, with a different constant of proportionality. This expression captures a sense in which more central agents receive stronger incentives, but $\widetilde{\boldsymbol{G}}^{-1}\mathbf{1}$ behaves quite differently from standard measures of centrality such as Katz–Bonacich centrality. In particular, the inverse $\boldsymbol{G}^{-1}$ changes non-monotonically as $\boldsymbol{G}$ changes. This can induce non-monotonicities in the optimal contract and the resulting actions and utilities. We next describe several comparative statics exercises that highlight some consequences of such non-monotonicities.

5.3. **Comparative statics.** In this section, we explore how the optimal contract, as well as the agents' and principal's payoffs, vary with the technology of production. The simple form of the team performance function Y in our environment, as well as the explicit characterization of incentives and outcomes, facilitate this exercise. We focus on the effects of two types of changes to the network: a change in an agent's local complementarities and a change in the overall strength of complementarities. Section 5.3.1 examines how the optimal team performance depends on the network of complementarities $\boldsymbol{G}$. The results demonstrate some interesting tensions between the principal's and the agents' interests. Section 5.3.2 then explores a practical question about compensation: how the total share of output optimally used for incentive pay depends on the strength of complementarities.

5.3.1. *Varying the network.* We look at how the principal's and agents' payoff vary as the network changes.

Proposition 5. *The principal's payoff is weakly increasing in the edge weight* $G_{ij} = G_{ji}$.

The principal obtains weakly higher profits from an increase in edge weights. However, it need not be the case that agents prefer such a perturbation. We will illustrate this through a network on three agents (see Figure 1); general comparative statics can be found in Appendix D.

Without loss of generality, we can assume $G_{12} \geq G_{13} \geq G_{23}$ and choose the normalization $G_{12} = 1$, so that the adjacency matrix is

$$\boldsymbol{G} = \begin{bmatrix} 0 & 1 & G_{13} \\ 1 & 0 & G_{23} \\ G_{13} & G_{23} & 0 \end{bmatrix}.$$

Figure 2 shows the optimal payments and the corresponding equilibrium payoffs as we vary the link weight G_{23}, under parameter values specified in the caption. Figure 2a depicts optimal payments to each agent as a function of G_{23}. The payment is non-monotonic in own links: once payments are nonconstant in the strength of that link, increasing G_{23} initially decreases agent 2's payment. The numerical example also illustrates a corresponding non-monotonicity in payoffs: strengthening one of an agent's links can decrease his equilibrium payoff under the optimal contract. Figure 2b depicts

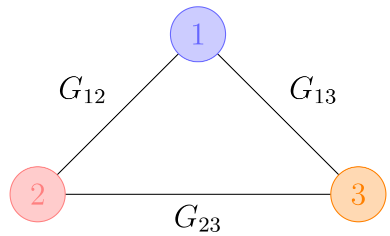


FIGURE 1. Three agent weighted graph with weights G_{12}, G_{13}, and G_{23}.

the equilibrium payoffs under optimal payments as a function of G_{23}. Strengthening the link between agents 2 and 3 can *decrease* the resulting payoffs for agents 1 and 2.

This finding contrasts with an intuition that one might have from the network games literature, that agents are better off from becoming more central. Under fixed payments, all agents' payoffs are monotone in the network. In the present setting, however, agent 2 can benefit from weakening one of his links.

There is therefore a tension between the network formation incentives of the principal and the agents. Agents may not be willing to form links that would benefit the principal or the team as a whole, even if link formation is not costly.

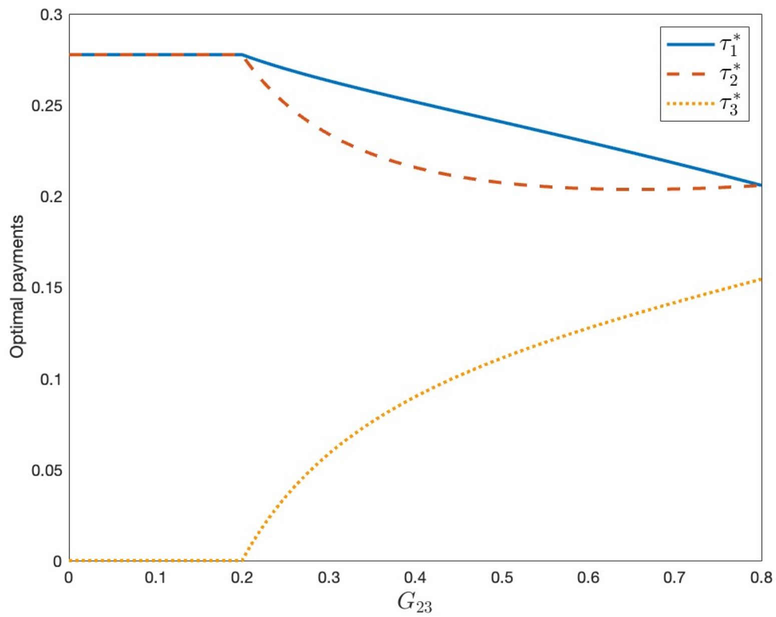
5.3.2. *Varying complementarities.* We now turn to how total payments change as the strength of complementarities increases. To operationalize this, we introduce a non-negative parameter β and set the network $\boldsymbol{G}$ to be

$$\boldsymbol{G} = \beta \widehat{\boldsymbol{G}}$$

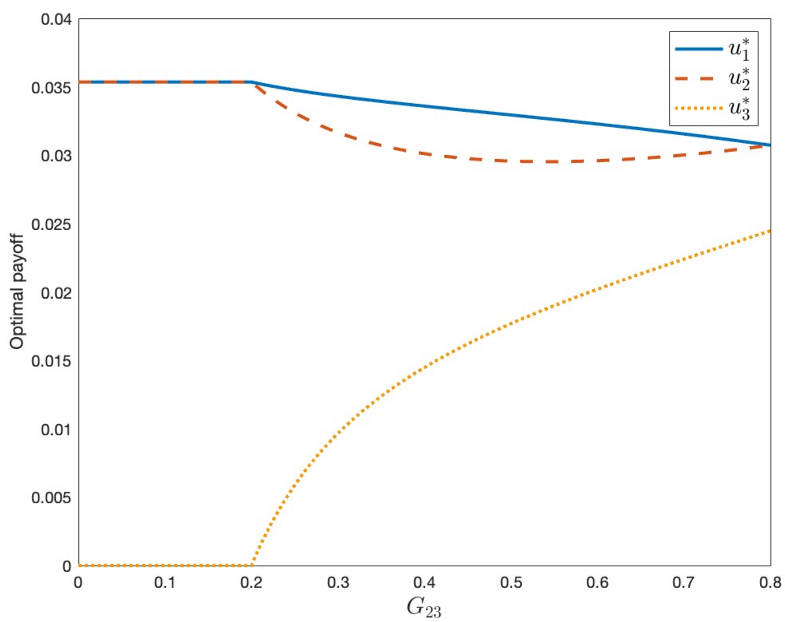
for some fixed network $\widehat{\boldsymbol{G}}$, which allows us to scale complementarities while fixing their relative levels. The production function in this representation is

$$Y(\boldsymbol{a}) = \sum_{i \in N} a_i + \frac{\beta}{2} \sum_{i,j \in N} \widehat{G}_{ij} a_i a_j.$$

We study the comparative static in the special case when $P(\cdot)$ is linear in the range of feasible team performance. We assume for simplicity that the optimal contract is unique, but could easily relax this assumption. The principal faces a trade-off between keeping a larger percentage of its value and using larger payments to encourage workers to exert more effort. The following result states that when complementarities in production are larger, it is optimal to keep a smaller percentage of a larger pie.



(A) Optimal payments



(B) Payoffs under optimal payments

FIGURE 2. The optimal payments and resulting equilibrium payoffs as a function of the weight G_{23}. Here $G_{13} = 0.8$ while $P(Y) = \min\{0.5Y, 1\}$ (the kink is not relevant for the principal's problem). In both diagrams, the curve corresponding to agent 1 is the topmost (solid blue) one; the curve corresponding to agent 2 is the second from the top (dashed red); and the curve corresponding to agent 3 is the lowest (dotted orange) one.

Proposition 6. *Suppose that $P(Y) = \kappa Y$ on an interval $[0, \overline{Y}]$ containing the equilibrium team performance under any contract and that there is a unique optimal contract $\boldsymbol{\tau}^*$. The sum of agents' payments under the optimal contract is increasing in the strength of complementarities β, i.e.,*

$$\frac{\partial \left(\sum_{i \in N} \tau_i^*\right)}{\partial \beta} > 0.$$

The basic idea behind the proof is that the benefits to retaining more of the firm are linear in the probability of success while the benefits to allocating more shares to workers are convex, and become steeper as complementarities increase.

If $P(Y)$ is strictly concave, there is a trade-off between the concavity of $P(Y)$ and the convexity of $Y(\boldsymbol{a})$. Depending on which effect is stronger, the total payments made to agents may increase or decrease as complementarities grow stronger.

In the rest of the paper, we analyze the example's original formulation which normalizes the complementarity parameter β to 1.

5.4. **Trade-off between marginal productivity and centrality.** The parametric setting we have been working with throughout this section has been quite special in that there was no heterogeneity in standalone productivity—i.e., each agent would have the same productivity if no other teammates exerted effort. This drove some of the specifics of how the optimal contract responded to changes in other parameters. Once we move beyond such a setting, by adding standalone productivity heterogeneity, new phenomena emerge. In particular, as we will now see, it is possible that agents' productivities and centralities must shift in opposite directions to keep the balance condition satisfied.

To formalize this, suppose that $\boldsymbol{\tau}^*$ is an optimal contract given team performance $Y(\boldsymbol{a})$ and probabilities of outcomes $P_s(Y)$ and $\widehat{\boldsymbol{\tau}}^*$ is an optimal contract given team performance $\widehat{Y}(\boldsymbol{a})$ and probabilities $\widehat{P}_s(Y)$.

Corollary 4. *Consider a pair of risk-neutral agents i and j, each receiving a positive payment at some common outcome s under $\boldsymbol{\tau}^*$ and $\widehat{\boldsymbol{\tau}}^*$. If $\frac{\alpha_i}{\alpha_j}$ is strictly higher (respectively, strictly lower) under contract $\boldsymbol{\tau}^*$ than $\widehat{\boldsymbol{\tau}}^*$, then $\frac{c_i}{c_j}$ is strictly lower (respectively, strictly higher) under contract $\boldsymbol{\tau}^*$ than $\widehat{\boldsymbol{\tau}}^*$.*

At the optimal contract, the balance condition in Theorem 1 must hold, that is,

$$\alpha_i c_i = \alpha_j c_j.$$

If the environment is perturbed so that the relative marginal productivity at the optimal contract $\frac{\alpha_i}{\alpha_j}$ strictly increases, then the relative centrality $\frac{c_i}{c_j}$ strictly decreases. This implication is notable, since contract optimality may require one's marginal productivity to respond in the opposite direction of one's centrality (where both are interpreted in relative terms).

This raises a question of whether $\frac{\alpha_i}{\alpha_j}$ can strictly increase. As we saw in Section 5.2, there are settings where $\frac{\alpha_i}{\alpha_j}$ and $\frac{c_i}{c_j}$ both remain constant as the environment changes. We now show that if we extend the example we have been working with to introduce heterogeneity in agents' standalone contributions to team performance, these quantities do indeed move in opposite directions.

Example 2. Our example is based on the paper's running example with two agents. Consider a network with adjacency matrix

$$\boldsymbol{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Suppose team performance is

$$Y(\boldsymbol{a}) = (1+\delta)a_1 + a_2 + a_1 a_2,$$

for a strictly positive δ. The principal observes whether the project succeeds and fails, with the probability of success given by a strictly increasing, concave and twice differentiable function $P(Y)$.

Fact 3. Consider any $\delta > 0$. At any optimal contract where both agents receive a payment, the marginal productivities satisfy $\alpha_1 > \alpha_2$.

Corollary 4 implies the centralities must satisfy $c_1 < c_2$. Under the principal's favorite contract, the agent with a higher standalone contribution to team performance has a higher marginal productivity but is less central in the endogenous network of spillovers.

The effect elucidated in this section complements our previous analysis. Recall that our main characterization of optimal incentives requires equalizing (across agents) a product of (i) marginal productivities α_i; (ii) total effect on team performance c_i; and (iii) expected marginal utility $u_i'(\cdot)$ evaluated at equilibrium payments. The analysis in Section 4.1 focused on the role of term (iii) in achieving balance, showing that agents with identical strictly concave utility functions can be ranked in terms of payments, based on who has the larger product $\alpha_i c_i$. In this section, we turned off that margin of adjustment, and the consequence is that balance may have to be achieved by reducing the centrality of those agents who become more productive.

6. Team design under optimal contracts

Our results in Section 5.2 characterized the optimal contract given an active set (the set of all active agents). We now briefly discuss the problem of determining which agents should be active. This discrete optimization problem requires new insights beyond the intensive margin analysis and further demonstrates applications of our balance condition.

We continue in the standard network games setting of Example 1. We have two main results, which imply considerable structure on the active set. Our first result shows that optimal active sets are always tightly connected:

Proposition 7. *The diameter*[13] *of the active set under any optimal contract is at most* 2*.*

This result implies that any two active agents either have complementarities with each other or both have complementarities with some shared active neighbor.

For unweighted networks, we can characterize the optimal active set even more precisely:

Proposition 8. *If* $\boldsymbol{G}$ *is an unweighted network, then any maximum clique*[14] *is the active set at an optimal contract.*

These results suggest that when a firm relies on a single joint outcome to provide incentives, teams with dense or tightly-knit complementarities outperform more dispersed teams. The principal may prefer to make a small team exert large efforts to make the

[13]Maximum distance between two agents.

[14]Complete subgraph.

most use of complementarities, rather than eliciting less effort from a larger group with more diffuse complementarities.

Economically, this preference for concentrated incentives arises from the structure of the moral hazard problem. When equity is given to members of a tight-knit group, the incentives given to one member also motivate effort by the others due to spillovers. In contrast, if a given amount of equity is divided between two subsets of agents that are not tightly linked, the equity given to one subset dilutes the incentives of the other without a strong counteracting beneficial effect of spillovers.

These findings have implications for organizational design, describing an important force pushing firms to seek teams with dense internal complementarities when relying on equity-based incentives tied to overall firm performance. Of course, with more steeply increasing marginal costs of effort, these conclusions would be suitably adjusted. The main thing we learn from these results is that the extensive margin problem seems interesting, and intensive-margin incentive design has implications for optimal team composition. At the opposite extreme, if one would prefer not to think about extensive margin issues, Appendix C presents some sufficient conditions under which optimal teams include all agents.

Mathematically, the key to both results is the following reduction:

Lemma 3. *A contract $\boldsymbol{\tau}$ is optimal among those with a given $\sum_{i \in N} \tau_i = s$ (sum of payments in case of success) if and only if it solves*

$$\begin{aligned} &\max_{\boldsymbol{\tau}} && \lambda \\ &\textit{subject to} && (\boldsymbol{G\tau})_i = \lambda \textit{ whenever } \tau_i > 0. \end{aligned} \tag{7}$$

This lemma allows us to reformulate the principal's problem as maximizing the constant λ in the balanced neighborhood equity condition. The proof leverages the fact that, under balanced equity, we can express team performance as an increasing function of λ.[15] The proof of Proposition 7 proceeds by contradiction: if there were two agents at distance greater than two, we could construct an improved allocation by concentrating shares on a pair of agents connected by a high-weight link.

[15]Specifically, $Y(\boldsymbol{a}^*) = (\sum_{i=1}^n \tau_i) \left(\frac{P'(Y^*)}{1-P'(Y^*)\lambda} + \frac{P'(Y^*)^2 \lambda}{2(1-P'(Y^*)\lambda)^2} \right)$.

The proof of Proposition 8 relies on a careful inductive construction. Given an arbitrary optimal allocation, we iteratively build a clique within the active set that achieves the same constant λ. This construction crucially uses the balanced equity condition at each step.

7. Optimal equity pay

The contracts we have described so far are finely tailored to individual outcomes (see Corollary 2). In practice, such contracts may be difficult to implement, and firms often use simple compensation schemes. Our results can be adapted to characterize optimal contracts within a restricted class. This section provides an illustration by analyzing one widely used incentive scheme: equity pay. Note that in simple success-or-failure environments, all optimal contracts feature equity pay, but in general the optimal equity contract need not match the optimal unrestricted contract.

An equity pay contract pays each agent a fixed share $\sigma_i v_s$ of the surplus v_s produced by the team. For a given equity contract $\boldsymbol{\sigma}$, the expected payoff to the principal is

$$\left(1 - \sum_{i \in N} \sigma_i\right) \sum_{s \in \mathcal{S}} v_s P_s(Y).$$

The expected payoff to agent i from an equity share σ_i is

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} u_i\left(\sigma_i v_s\right) P_s(Y) - c_i(a_i).$$

The result below characterizes an optimal equity contract $\boldsymbol{\sigma}^*$. We maintain Assumption 1, which now states that there is a neighborhood of $\boldsymbol{\sigma}^*$ in the space of equity contracts where $\boldsymbol{\sigma}(\boldsymbol{a}^*)$ is continuously differentiable.

Proposition 9. *Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract and Y^* is the induced team performance. There exists a constant λ such that for any agent i receiving a positive equity payment, we have*

$$\alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) v_s u_i'(\sigma_i^* v_s) = \lambda.$$

The proof follows a similar approach to the proof of Theorem 1. It involves analyzing the effect of perturbations to equity payments on the principal's objective. Perturbations in the equity payment of an agent affect payments at all outcomes. The direct effect of increasing σ_i on i's action is proportional to the change in marginal expected utility from payments, which is given by the expression

$$\alpha_i \sum_{s \in \mathcal{S}} P_s'(Y^*) v_s u_i' \left(\sigma_i^* v_s\right).$$

The summation captures the total direct effect of increasing an agent's equity on team performance by aggregating across outcomes, and multiplying by c_i includes indirect effects. At an optimal equity contract, the effect of perturbing equity payments on total team performance must be the same for all agents with positive equity.

In general, the balance condition in Proposition 9 characterizing optimal equity contracts does not match the condition in Section 3 characterizing optimal contracts. An optimal contract fine-tunes payments at each outcome, incentivizing agents to exert optimal effort levels. Equity pay imposes a particular linear relationship between the payments for different outcomes that may be practically convenient but sacrifices some incentive power. We note the contrast with Dai and Toikka (2022), which finds that linear contracts are optimal for a principal designing team incentives that must be robust to uncertainty about the environment.

8. Concluding discussion

We have studied an incentive design problem for a team whose members contribute via unobserved effort. We investigate how optimal contracts depend on the team's production function. Our main contribution is a necessary condition for contract optimality. We show that optimal contracts must satisfy a balance condition across agents receiving positive incentive pay.

The balance result in Theorem 1 generalizes prior work on complementarities in optimal contract design. Beyond the lack of parametric assumptions, our general necessary condition does not require that strategic interactions take any particular form (such as strategic complementarities). What is key to our analysis is studying the perturbations of equilibria, and this is possible with general spillovers. Specific assumptions on spillover structure can, however, be very helpful for guaranteeing equilibrium existence.

While we have focused on a principal's optimization, the analysis of how contract perturbations affect team performance is equally relevant for understanding other design problems, including cases where there is no principal. The key insight that our exercises demonstrate is that, under suitable conditions, the first-order conditions of such an optimization problems involve only the local comparative statics of the equilibrium. These can be exploited for useful characterizations of optimal designs. We leave fleshing out these implications to future work.

An important simplifying assumption throughout the analysis is that actions influence outcomes only via a one-dimensional team performance $Y(\boldsymbol{a})$. Moving beyond this assumption to settings where probabilities of outcomes depend in an arbitrary way on the full action profile, a first-order approach may continue to be fruitful, in that we expect that generalizations of our balance conditions would hold. But the outcome distribution may now provide more fine-grained information about an agent's effort, and the principal will use this information to design optimal incentives (as in, e.g., Holmström (1982) and Legros and Matthews (1993)). The generalized balance conditions must incorporate this along with agents' centralities and marginal productivities. So the balance conditions would be suitably adjusted, and rankings such as those in Section 4 would also be affected by these informational considerations.

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Appendix A. Omitted proofs

A.1. **Proof of Fact 1.** For contracts in this compact space, agent best responses will also be contained in a compact space. We want to show the supremum π of attainable principal contracts is attainable. Choose a sequence of contracts $\boldsymbol{\tau}^{(i)}$ giving payoffs converging to π and let $(\boldsymbol{a}^{*})^{(i)}$ be the corresponding equilibria.[16] By compactness, we can choose a subsequence of $\boldsymbol{\tau}^{(i)}$ such that $\boldsymbol{\tau}^{(i)} \to \boldsymbol{\tau}^*$ and $(\boldsymbol{a}^*)^{(i)} \to \boldsymbol{a}^*$. By upper-hemicontinuity of equilibrium, the action profile $\boldsymbol{a}^*$ is an equilibrium under $\boldsymbol{\tau}^*$. Since the principal's payoffs are continuous in the contract and actions, the contract $\boldsymbol{\tau}^*$ attains the optimal payoff π.

A.2. **Proof of Lemma 1.** We begin by observing that under any optimal contract,

$$a_i^*(\boldsymbol{\tau}^*) = 0 \iff \tau_i^*(s) = 0, \quad \text{for all } s \in \mathcal{S}.$$

That is, at an optimal contract, an agent i exerts zero effort at equilibrium, if and only if it does not receive a payment from the contract at any outcome.[17]

We analyze the change in team performance as the transfers to agents are perturbed. Consider contract $\boldsymbol{\tau}$ and any agent i for which there exists an outcome s' such that $\tau_i(s') > 0$. For any outcome s, consider marginally increasing $\tau_i(s)$. The change induced by this perturbation is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\boldsymbol{a}^*)^T \cdot \frac{\partial \boldsymbol{a}^*}{\partial \tau_i(s)}, \tag{8}$$

where $\boldsymbol{a}^*$ is the equilibrium action profile for the contract $\boldsymbol{\tau}$. The substance of the proof is analyzing the second term on the right-hand side of (8).

[16] If π is zero, then the contract giving zero payments under all outcomes is optimal. So we can assume that π is positive and thus an equilibrium exists under $\boldsymbol{\tau}^{(i)}$ for i sufficiently large.

[17] Suppose at an optimal contract $\boldsymbol{\tau}^*$, there is an agent i who receives positive payment $\tau_i^*(s) > 0$ under an outcome s but chooses action $a_i^*(\boldsymbol{\tau}^*) = 0$. Then the principal receives a strictly higher payment under the contract $\boldsymbol{\tau}^\dagger$ which sets $\tau_i^\dagger(s) = 0$ and is otherwise equal to $\boldsymbol{\tau}^*$. At this contract, agent i chooses action $a_i = 0$ for any profile of actions $\boldsymbol{a}_{-i}$ played by other agents. Thus, the equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$ under contract $\boldsymbol{\tau}^*$ is also an equilibrium profile under contract $\boldsymbol{\tau}^\dagger$. Since outcome s occurs with positive probability under any team performance, the principal's expected payments to agents are strictly higher under $\boldsymbol{\tau}^*$ than $\boldsymbol{\tau}^\dagger$. The other direction is straightforward.

First, consider any agent j such that $a_j^*(\boldsymbol{\tau}) = 0$. The change in their equilibrium action due to an increase in $\tau_i(s)$ is zero. The utility to agent j at contract $\boldsymbol{\tau}$ is

$$\mathcal{U}_j = \sum_{s' \in \mathcal{S}} P_{s'}(Y^*) u_j(\tau_j(s')) - C_j(a_j).$$

At contract $\boldsymbol{\tau}$, agent j receives no payment under any outcome, so has a unique best response of $a_j^* = 0$.

It is thus without loss to analyze the change in equilibrium actions of agents j that take a strictly positive action in profile $\boldsymbol{a}^*$, that is, $a_j^* > 0$. The analysis from here on focuses on such agents, overloading notation to represent the actions of these agents by $\boldsymbol{a}^*$.

We will show that the change in equilibrium actions $\boldsymbol{a}^*$ as the transfer $\tau_i(s)$ increases is

(9)

$$\frac{\partial \boldsymbol{a}^*}{\partial \tau_i(s)} = \boldsymbol{H}^{-\frac{1}{2}} \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}} \boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1} \boldsymbol{H}^{-\frac{1}{2}} \begin{bmatrix} \boldsymbol{0} \\ \frac{\partial Y}{\partial a_i} P_s'(Y) u_i'(\tau_i(s)) \\ \boldsymbol{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} \left[\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}\right]^{-1} \boldsymbol{d},$$

for some vector $\boldsymbol{d}$.

Consider the equilibrium action profile $\boldsymbol{a}^*$. For an agent j, the first-order conditions imply a_j^* must solve the equation

$$C_j'(a_j) = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s')) \right) \frac{\partial Y}{\partial a_j}. \tag{10}$$

To arrive at (9), let us implicitly differentiate (10) with respect to $\tau_i(s)$. The resulting expression depends on the identity of agent j in comparison to i, the agent whose payment is perturbed. For all $j \neq i$,

$$C_j''(a_j^*) \frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s')) \right) \left(\sum_{k=1}^{n} \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i(s)} \right) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i(s)} \cdot \sum_{s' \in \mathcal{S}} P_{s'}''(Y) u_j(\tau_j(s')). \tag{11}$$

On the other hand, for $j = i$,

$$C_j''(a_j^*)\frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s'\in\mathcal{S}} P_{s'}'(Y)u_j(\tau_j(s'))\right)\left(\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial a_k \partial a_j}\cdot\frac{\partial a_k^*}{\partial \tau_i(s)}\right) + \frac{\partial Y}{\partial a_j}P_s'(Y)u_j'(\tau_j(s)) + \frac{\partial Y}{\partial a_j}\cdot\frac{\partial Y}{\partial \tau_i(s)}\sum_{s'\in\mathcal{S}}P_{s'}''(Y)u_j(\tau_j(s')). \tag{12}$$

We can combine (11) and (12) to write the resulting expression in vector form below

$$\frac{\partial \boldsymbol{a}^*}{\partial \tau_i(s)} = [\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\begin{bmatrix}\mathbf{0}\\ \frac{\partial Y}{\partial a_i}P_s'(Y)u_i'(\tau_i(s))\\ \mathbf{0}\end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)}[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d},$$

where $\boldsymbol{d}$ is a vector with j^{th} element defined as

$$d_j := \frac{\partial Y}{\partial a_j}\cdot\sum_{s'\in\mathcal{S}}P_{s'}''(Y)u_j(\tau_j(s')).$$

The expression in (9) follows.

Substituting (9) into (8), the change in team performance as the transfer $\tau_i(s)$ increases is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\boldsymbol{a}^*)^T\boldsymbol{H}^{-\frac{1}{2}}\left[\boldsymbol{I}-\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1}\boldsymbol{H}^{-\frac{1}{2}}\begin{bmatrix}\mathbf{0}\\ \frac{\partial Y}{\partial a_i}P_s'(Y)u_i'(\tau_i(s))\\ \mathbf{0}\end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)}\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}.$$

Applying the definitions of α_i and c_i, we obtain

$$\frac{\partial Y}{\partial \tau_i(s)} = \alpha_i c_i P_s'(Y)u_i'(\tau_i(s)) + \frac{\partial Y}{\partial \tau_i(s)}\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}.$$

Rearranging,

$$\frac{\partial Y}{\partial \tau_i(s)} = \frac{1}{1-\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}}\cdot\alpha_i c_i P_s'(Y)u_i'(\tau_i(s)).$$

Setting $l = \frac{1}{1-\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}}$ and observing l does not depend on i, we obtain the desired result.

A.3. **Proof of Theorem 1.** The expected payoff for the principal under contract $\boldsymbol{\tau}$ and corresponding equilibrium actions $\boldsymbol{a}^*$ is

$$\sum_{s' \in \mathcal{S}} \left(v_{s'} - \sum_{i \in N} \tau_i(s') \right) P_{s'}(Y(\boldsymbol{a}^*)).$$

Suppose $\boldsymbol{\tau}^*$ is an optimal contract inducing equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$ with team performance Y^*. Consider outcome s and any agent i such that $\tau_i^*(s) > 0$. Then the first-order condition for $\tau_i^*(s)$ implies that

$$\frac{dY}{d\tau_i(s)} \underbrace{\sum_{s' \in \mathcal{S}} \left(v_{s'} - \sum_{i \in N} \tau_i^*(s') \right) P'_{s'}(Y^*)}_{D} = P_s(Y^*).$$

The left-hand side is the benefit from increasing $\tau_i^*(s)$ while the right-hand side is the expected additional transfer required. Since $P_s(Y^*) > 0$ by assumption, the summation labeled D is nonzero.

Substituting Lemma 1 in the above equation, we obtain

$$l\alpha_i c_i P'_s(Y^*) u'_i(\tau_i^*(s)) = \frac{P_s(Y^*)}{D},$$
$$\iff \alpha_i c_i u'_i(\tau_i^*(s)) = \lambda_s,$$

where $\lambda_s = P_s(Y^*)/(lP'_s(Y^*)D)$. Observing that λ_s is independent of i, the statement of the result follows.

A.4. **Proof of Corollary 1.** Let $\mathcal{S}_{ij}^*$ have at least 2 outcomes. (If $|\mathcal{S}_{ij}^*| \leq 1$, the statement holds vacuously.) By Theorem 1, for any $s \in \mathcal{S}_{ij}^*$, there is a constant $\lambda_s \neq 0$ such that

$$\alpha_i c_i u'_i(\tau_i^*(s)) = \lambda_s, \quad \text{and} \quad \alpha_j c_j u'_j(\tau_j^*(s)) = \lambda_s.$$

It follows that

$$\frac{u'_i(\tau_i^*(s))}{u'_j(\tau_j^*(s))} = \frac{\alpha_j c_j}{\alpha_i c_i}.$$

The right-hand side is independent of s, so the result follows with λ_{ij} equal to this right-hand side.

A.5. **Proof of Proposition 1.** We prove a couple of lemmas which help in proving the proposition statement. The first lemma gives a condition which must hold for all outcomes at which an agent receives a positive payment.

Lemma 4. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and Y^* is the induced team performance. For all s in the set of outcomes $\mathcal{S}_i^*$ where i receives a positive payment, $P_s'(Y^*) > 0$.*

Proof. Consider agent i and let $\mathcal{S}_i^*$ be the set of outcomes at which agent i receives a positive payment. If $\mathcal{S}_i^*$ is the empty set, the result holds vacuously. Otherwise, we will show that, either

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*, \quad \text{or,} \quad P_s'(Y^*) < 0, \text{ for all } s \in \mathcal{S}_i^*. \tag{13}$$

Recall from the proof of Theorem 1 that

$$l\alpha_i c_i P_s'(Y^*) u_i'(\tau_i^*(s)) = \frac{P_s(Y^*)}{\sum_{s' \in \mathcal{S}} \left(v_{s'} - \sum_{i \in N} \tau_i(s')\right) P_{s'}'(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Since every outcome occurs with non-zero probability, it must be that $P_s'(Y^*) \neq 0$ for any outcome in $\mathcal{S}_i^*$. In the case $\mathcal{S}_i^*$ has exactly 1 outcome, it follows that $P_s'(Y^*) > 0$ or $P_s'(Y^*) < 0$ for $s \in \mathcal{S}_i^*$. Thus, suppose $\mathcal{S}_i^*$ has at least 2 outcomes. Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

Since the utility function $u_i(\cdot)$ is strictly increasing, we must have either

$$P_s'(Y^*) > 0 \text{ for } s \in \{s_1, s_2\}, \quad \text{or,} \quad P_s'(Y^*) < 0 \text{ for } s \in \{s_1, s_2\}.$$

The statement in (13) follows. We now show that

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*.$$

The equilibrium condition for agent i is

$$C_i'(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P_s'(Y^*) u_i(\tau_i^*(s)).$$

Since $\sum_{s\in\mathcal{S}} P_s'(Y) = 0$, the equilibrium condition can be rewritten as

$$C_i'(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s\in\mathcal{S}_i^*} P_s'(Y^*)\left(u_i(\tau_i^*(s)) - u_i(0)\right).$$

By assumption we have a positive marginal productivity, that is, $\frac{\partial Y}{\partial a_i} > 0$. The cost of effort is strictly increasing, that is, $C_i'(\cdot) > 0$. The utility function $u_i(\cdot)$ is strictly increasing in payments. Thus,

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*$$

as desired. □

The second lemma shows the existence of a common outcome at which agents receiving a positive payment are paid.

Lemma 5. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract. Consider a pair of agents i and j, each with strictly concave utility functions. If there exist outcomes s_i and s_j such that $\tau_i^*(s_i) > 0$ and $\tau_j^*(s_j) > 0$, then there exists an outcome $s \in \mathcal{S}$ such that*

$$\tau_i^*(s) > 0 \text{ and } \tau_j^*(s) > 0.$$

Proof. Suppose there does not exist an outcome at which both agents receive a positive payment. Thus, the payments $\tau_i^*(s_j) = 0$ and $\tau_j^*(s_i) = 0$. The KKT first-order conditions at optimal contract $\boldsymbol{\tau}^*$ are

$$lD\alpha_k c_k u_k'(\tau_k^*(s_k))P_{s_k}'(Y^*) - P_{s_k}(Y^*) = 0 \quad \text{for } k \in \{i,j\}. \tag{14}$$

In addition to the above set of equations, we also have

$$lD\alpha_k c_k u_k'(0)P_{s_{\{i,j\}\setminus k}}'(Y^*) - P_{s_{\{i,j\}\setminus k}}(Y^*) \le 0 \quad \text{for } k \in \{i,j\}. \tag{15}$$

Recall $\mathcal{S}_i^*$ is the set of outcomes where agent i receives a positive payment under contract $\boldsymbol{\tau}^*$.

Since $P_s'(Y^*) > 0$ for any $s \in \mathcal{S}_i^*$ (see Lemma 4), we must have $lD\alpha_i c_i > 0$. Consider the following chain of inequalities for agent i:

$$\frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)} < lD\alpha_i c_i u_i'(0) \le \frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)}. \tag{16}$$

Both the inequalities follow from applying (14) and (15) to agent i. We utilize the fact that $u_i(\cdot)$ is strictly concave. We also utilize the observation that, since agent i and j receive a positive payment at outcome s_i and s_j, Lemma 4 tells us that $P'_{s_i}(Y^*) > 0$ and $P'_{s_j}(Y^*) > 0$. Following the same computation for agent j, we obtain the inequalities

$$\frac{P_{s_j}(Y^*)}{P'_{s_j}(Y^*)} < lD\alpha_j c_j u'_j(0) \leq \frac{P_{s_i}(Y^*)}{P'_{s_i}(Y^*)}. \tag{17}$$

This contradicts inequality (16). Thus, if two agents receive a positive payment at some (potentially different) outcomes under the optimal contract, then there must exist an outcome at which both agents receive a positive payment. □

Proof of Proposition 1. Consider agents i and j with identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$. The statement trivially holds if either agent i or agent j receives a 0 payment at all outcomes. Thus, consider a scenario where there exist outcomes s_i and s_j such that

$$\tau_i^*(s_i) > 0 \text{ and } \tau_j^*(s_j) > 0.$$

By Lemma 5, it suffices to show that when there exists an outcome such that both agents i and j receive a positive payment at this outcome, then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

Let $\mathcal{S}_{ij}^*$ be the set of outcomes at which both agents receive a positive payment under contract $\boldsymbol{\tau}^*$. The set $\mathcal{S}_{ij}^*$ is non-empty. We can assume without loss of generality that $\tau_i^*(s) \geq \tau_j^*(s)$ for some outcome $s \in \mathcal{S}_{ij}^*$. We show that then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S}.$$

Applying Corollary 1, it holds that

$$|\alpha_i c_i| \geq |\alpha_j c_j|.$$

Additionally, $\alpha_i c_i$ and $\alpha_j c_j$ are either both positive or negative. Further applying Corollary 1 to any outcome $s' \in \mathcal{S}^*_{ij}$, the ratio of marginal utilities satisfies

$$\frac{u_i'(\tau_i^*(s'))}{u_j'(\tau_j^*(s'))} = \frac{\alpha_j c_j}{\alpha_i c_i} \leq 1.$$

This implies that agent i receives a weakly larger payment than agent j under all outcomes in $\mathcal{S}^*_{ij}$, that is,

$$\tau_i^*(s) \geq \tau_j^*(s) \quad \textit{for all } s \in \mathcal{S}^*_{ij}.$$

We will show that this ordering on payments holds for outcomes in the set $\mathcal{S} \setminus \mathcal{S}^*_{ij}$ as well. The ordering trivially holds at outcomes where $\tau_j^*(s) = 0$. Consider an outcome s at which $\tau_j^*(s) > 0$ but $\tau_i^*(s) = 0$. We show that such an outcome cannot exist at an optimal contract $\boldsymbol{\tau}^*$. We showed in the proof of Theorem 1 that the first-order condition for the principal is

$$lD\alpha_j c_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*) = 0.$$

Since the utility to agent j is strictly increasing, it must hold that

$$lD\alpha_j c_j P_s'(Y^*) > 0.$$

Recall, $|\alpha_i c_i| \geq |\alpha_j c_j|$ and they are either both positive or negative. Using the fact that $u_i(\cdot)$ and $u_j(\cdot)$ are strictly increasing identical utility functions, and thus $u_i'(0) = u_j'(0) > 0$, we conclude

$$lD\alpha_i c_i u_i'(0)P_s'(Y^*) \geq lD\alpha_j c_j u_j'(0)P_s'(Y^*). \tag{18}$$

Now, consider the following chain of inequalities:

$$\begin{aligned} lD\alpha_i c_i u_i'(0)P_s'(Y^*) - P_s(Y^*) &\geq lD\alpha_j c_j u_j'(0)P_s'(Y^*) - P_s(Y^*), \\ &> lD\alpha_j c_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*), \\ &= 0. \end{aligned}$$

The first inequality follows from (18). The second inequality follows from the fact that $u_j(\cdot)$ is strictly concave and $lD\alpha_j c_j P_s'(Y^*) > 0$. The left-hand side in the above chain of inequalities is the derivative of the principal's objective in $\tau_i(s)$. The derivative being positive contradicts the optimality of $\boldsymbol{\tau}^*$, so the statement of the proposition holds. □

A.6. **Proof of Corollary 2.** Recall from the proof of Theorem 1 that

$$l\alpha_i c_i P_s'(Y^*) u_i'(\tau_i^*(s)) = \frac{P_s(Y^*)}{\sum_{s'\in\mathcal{S}}\left(v_{s'} - \sum_{i\in N}\tau_i(s')\right)P_{s'}'(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

The statement is proved.

A.7. **Proof of Proposition 2.** For a risk-neutral agent i, the marginal value of money is constant: $u_i'(\cdot) = a_i$ for some $a_i > 0$. If at the optimal contract, the agent receives a positive payment under two outcomes s_1 and s_2 that occur with positive probability at Y^*, Corollary 2 implies

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = 1 \neq \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

We assumed the right hand side is not equal to 1 in the statement of Proposition 2. This gives a contradiction, so agent i can receive a positive payment in at most one outcome.

It remains to show all risk-neutral agents receive a payment at the same outcome. The arguments are essentially the same as those used to prove Lemma 5. Consider risk-neutral agents i and j. Suppose that agent i receives a positive payment at outcome s_i while agent j receives a positive payment at a distinct outcome s_j. Applying the arguments in Lemma 5, the inequality obtained for agent i is

$$\frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)} \leq l\alpha_i c_i u_i'(0) \leq \frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)}.$$

Similarly, the inequality obtained for agent j is

$$\frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)} \leq l\alpha_i c_i u_i'(0) \leq \frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)}.$$

These inequalities imply $\frac{P_{s_i}(Y^*)}{P'_{s_i}(Y^*)} = \frac{P_{s_j}(Y^*)}{P'_{s_j}(Y^*)}$, again contradicting our assumption these values are distinct. Thus, it must be that the risk-neutral agents receive a positive payment at the same outcome.

A.8. **Proof of Proposition 3.** Fixing shares $\boldsymbol{\tau}$ and others' strategies, agent i's expected payoff is strictly concave in his action a_i because $Y(\boldsymbol{a})$ is linear in a_i, the success probability $P(Y)$ is concave in Y, and the effort cost is strictly convex. So agent i has a unique best response, meaning we need only consider pure-strategy equilibria. Moreover marginal costs at $a_i = 0$ are zero while marginal benefits at $a_i = 0$ are strictly positive if $\tau_i > 0$ and zero if $\tau_i = 0$. Since U_i is concave in a_i, this rules out a boundary solution where the first-order condition $\frac{\partial U_i}{\partial a_i} = 0$ is not satisfied. So the first-order condition is necessary and sufficient for a best-response.

It follows that the following equations are necessary and sufficient for the vector $\boldsymbol{a}^*$ to be a Nash equilibrium:

$$[\boldsymbol{I} - P'(Y^*)\boldsymbol{TG}]\boldsymbol{a}^* = P'(Y^*)\boldsymbol{\tau} \text{ and } Y^* = Y(\boldsymbol{a}^*).$$

Given a constant y such that $P'(y)\rho(\boldsymbol{TG}) \neq 1$, where $\rho(\boldsymbol{TG})$ is the spectral radius of $\boldsymbol{TG}$, we can define actions by

$$\boldsymbol{a}^*(y) = [\boldsymbol{I} - P'(y)\boldsymbol{TG}]^{-1}P'(y)\boldsymbol{\tau}.$$

Solutions of the first-order conditions then correspond to solutions to

$$Y(\boldsymbol{a}^*(y)) = y.$$

The function $Y(\boldsymbol{a}^*(y))$ is strictly increasing in each coordinate of $\boldsymbol{a}^*(y)$. We analyze how $\boldsymbol{a}^*(y)$ changes as y increases. Consider the set

$$y_R := \{y : P'(y)\rho(\boldsymbol{TG}) < 1\}.$$

Observe that because $P(\cdot)$ is concave, if $y \in y_R$ then $y + \epsilon \in y_R$ for any $\epsilon > 0$. We show that constrained to the set y_R, there exists a unique fixed point to the function $Y(\boldsymbol{a}^*(y))$. Each coordinate of $\boldsymbol{a}^*(y)$ is weakly decreasing in y since $P'(\cdot)$ is weakly decreasing (by our assumption $P(\cdot)$ is concave). So $Y(\boldsymbol{a}^*(y))$ is decreasing, meaning there is at most one solution to $Y(\boldsymbol{a}^*(y)) = y$. It remains to show a solution to this equation exists.

We claim that we can find y such that $Y(\boldsymbol{a}^*(y)) \geq y$ and $P'(y)\rho(\boldsymbol{TG}) < 1$. If $P'(0)\rho(\boldsymbol{TG}) < 1$, the claim holds with $y = 0$ since $Y(\boldsymbol{a}^*(0)) \geq 0$. Otherwise, define y_0 by $P'(y_0)\rho(\boldsymbol{TG}) = 1$. A solution to this equation exists since $P'(y)$ is continuous and converges to zero as $y \to \infty$. Then $Y(\boldsymbol{a}^*(y)) \to \infty$ as $y \to y_0$ from above, so we have $Y(\boldsymbol{a}^*(y_0+\epsilon)) \geq y_0+\epsilon$ for $\epsilon > 0$ sufficiently small. This completes the proof of the claim.

Since $Y(\boldsymbol{a}^*(y))$ is decreasing in y, we can also choose y large enough such that $y > Y(\boldsymbol{a}^*(y))$. Since $Y(\boldsymbol{a}^*(y))$ is continuous in y, by the intermediate value theorem this function has a fixed point, denoted by y^*. We conclude that there exists a unique solution to $Y(\boldsymbol{a}^*(y)) = y$ in the set y_R and a corresponding profile $\boldsymbol{a}^*$ of equilibrium actions.

It remains to show that there does not exist an equilibrium $\boldsymbol{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\rho(\boldsymbol{TG}) \geq 1$. The case $\boldsymbol{\tau} = 0$ is immediate as the only equilibrium is $\boldsymbol{a}^* = 0$. Take $\boldsymbol{\tau}$ not identically zero and suppose there exists an equilibrium $\boldsymbol{a}^*$ such that $P'(Y^*)\rho(\boldsymbol{TG}) \geq 1$. It must solve the necessary and sufficient conditions

$$[\boldsymbol{I} - P'(Y^*)\boldsymbol{TG}]\boldsymbol{a}^* = P'(Y^*)\boldsymbol{\tau} \text{ and } Y^* = Y(\boldsymbol{a}^*). \tag{19}$$

By the Perron-Frobenius theorem,[18] there exists a left-eigenvector $\boldsymbol{v}$ of the matrix $P'(Y^*)\boldsymbol{TG}$ such that $\boldsymbol{v}$ has strictly positive entries. Multiplying the LHS of (19) by the vector v, we get

$$\begin{aligned}\boldsymbol{v}^T[\boldsymbol{I} - P'(Y^*)\boldsymbol{TG}]\boldsymbol{a}^* &= [1 - P'(Y^*)\rho(\boldsymbol{TG})]\boldsymbol{v}^T\boldsymbol{a}^* \\ &\leq 0,\end{aligned}$$

where the inequality follows from the assumption $P'(Y^*)\rho(\boldsymbol{TG}) \geq 1$ and the fact that $\boldsymbol{a}^*$ has strictly positive elements. However, we also compute

$$\begin{aligned}\boldsymbol{v}^T[\boldsymbol{I} - P'(Y^*)\boldsymbol{TG}]\boldsymbol{a}^* &= \boldsymbol{v}^T P'(Y^*)\boldsymbol{\tau} \text{ by (19)} \\ &> 0,\end{aligned}$$

[18]For this argument, it is without loss to assume the matrix $\boldsymbol{TG}$ is irreducible. If not, since $\boldsymbol{G}$ is symmetric, we can rewrite $\boldsymbol{TG}$ in a block diagonal form with irreducible blocks. Then $P'(Y^*)\rho(\boldsymbol{TG})$ must be an eigenvalue of at least one block of the matrix $P'(Y^*)\boldsymbol{TG}$. We can drop agents in all other blocks and apply the remainder of the argument to this block.

where the inequality holds because the entries of $\boldsymbol{v}$ are all positive and the entries of $\boldsymbol{\tau}$ are all non-negative and not identically zero. This is a contradiction, so there does not exist an equilibrium $\boldsymbol{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\rho(\boldsymbol{TG}) \geq 1$. We conclude the equilibrium described above is the unique one.

A.9. **Proof of Lemma 2.** By Theorem 1, we have that

$$\alpha_i c_i \text{ is constant across agents } i.$$

Suppose that there exist two agents $i^* \in N$ with $i^* = \arg\min_{k\in N} \alpha_k$ and $j^* \in N$ with $j^* = \arg\max_{k\in N} \alpha_k$ such that $\alpha_{i^*} < \alpha_{j^*}$.[19]

Then we have that, for agent i^*,

$$\alpha_{i^*} c_{i^*} < \alpha_{i^*}\alpha_{j^*} \sum_{j\in N} [\boldsymbol{I} - P'(Y^*)\boldsymbol{GT}]^{-1}_{i^*j} = (\alpha_{i^*})^2 \alpha_{j^*}, \tag{20}$$

using the maximality of α_{j^*} among the α_j and the definitions of c_{i^*} and α_{i^*}. But we similarly have that, for agent j^*,

$$\alpha_{j^*} c_{j^*} > \alpha_{j^*}\alpha_{i^*} \sum_{i\in N} [\boldsymbol{I} - P'(Y^*)\boldsymbol{GT}]^{-1}_{j^*i} = \alpha_{i^*} (\alpha_{j^*})^2. \tag{21}$$

Theorem 1 implies that $\alpha_{i^*} c_{i^*} = \alpha_{j^*} c_{j^*}$ for any two agents i^* and j^*, and so combining (20) and (21) implies

$$(\alpha_{i^*})^2 \alpha_{j^*} > \alpha_{i^*} (\alpha_{j^*})^2.$$

This contradicts our assumption $\alpha_{j^*} > \alpha_{i^*}$, so we must have α_i equal to some constant λ_1 for all i in N.

A.10. **Proof of Proposition 5.** Suppose $\boldsymbol{\tau}^*$ is an optimal contract for network $\boldsymbol{G}$ with equilibrium team performance $Y^*(\boldsymbol{G}, \boldsymbol{\tau}^*)$. Consider a perturbed network $\widetilde{\boldsymbol{G}}$ generated by increasing edge weight G_{ij} to $G_{ij} + \epsilon$ for some $\epsilon > 0$. We will show that contract $\boldsymbol{\tau}^*$ performs weakly better on network $\widetilde{\boldsymbol{G}}$ than on $\boldsymbol{G}$. Since we will be comparing $\boldsymbol{\tau}^*$ across networks, we suppress the dependence of equilibrium team performance on the contract.

[19] We are grateful to Michael Ostrovsky for suggesting the argument in the next paragraph.

Consider contract $\boldsymbol{\tau}^*$ and network $\widetilde{\boldsymbol{G}}$. We want to show that the equilibrium team performance $Y^*(\widetilde{\boldsymbol{G}})$ is at least $Y^*(\boldsymbol{G})$. The equilibrium actions solve

$$\boldsymbol{a}^*(\widetilde{\boldsymbol{G}}) = P'(Y^*(\widetilde{\boldsymbol{G}}))\left[\boldsymbol{I} - P'(Y^*(\widetilde{\boldsymbol{G}}))\boldsymbol{T}^*\widetilde{\boldsymbol{G}}\right]^{-1}\boldsymbol{\tau}^*.$$

Suppose $Y^*(\widetilde{\boldsymbol{G}}) < Y^*(\boldsymbol{G})$. It follows that $\boldsymbol{a}^*(\widetilde{\boldsymbol{G}})$ is point-wise strictly greater than $\boldsymbol{a}^*(\boldsymbol{G})$, because of the concavity of $P(\cdot)$ and the fact that $\widetilde{\boldsymbol{G}}$ is point-wise weakly greater than $\boldsymbol{G}$. However, this is a contradiction to $Y^*(\widetilde{\boldsymbol{G}}) < Y^*(\boldsymbol{G})$ because

$$Y(\boldsymbol{a}, \boldsymbol{G}) = \sum_i a_i + \frac{1}{2}\sum_{i,j} G_{ij}a_i a_j.$$

Thus, we must have $Y^*(\widetilde{\boldsymbol{G}}) \geq Y^*(\boldsymbol{G})$. The profits to the principal under contract $\boldsymbol{\tau}^*$ are thus weakly higher on network $\widetilde{\boldsymbol{G}}$ than on network $\boldsymbol{G}$:

$$\left(1 - \sum_i \tau_i^*\right) P\left(Y^*(\widetilde{\boldsymbol{G}})\right) \geq \left(1 - \sum_i \tau_i^*\right) P\left(Y^*\left(\boldsymbol{G}\right)\right).$$

Finally, the optimal contract for network $\widetilde{\boldsymbol{G}}$ must deliver at least as high a payoff as contract $\boldsymbol{\tau}^*$ does on network $\widetilde{\boldsymbol{G}}$.

A.11. **Proof of Proposition 6.** We can assume without loss of generality that all agents in $\widehat{\boldsymbol{G}}$ are active under $\boldsymbol{\tau}^*$ (by dropping any inactive agents from the network). Consider a feasible contract $\boldsymbol{\tau}$ satisfying the balanced neighborhood equity condition $\widehat{\boldsymbol{G}}\boldsymbol{\tau} = \lambda\mathbf{1}$ and let $s = \sum_i \tau_i \in [0,1]$ be the sum of shares under this contract. Such a contract will exist for any $s \in [0,1]$, as $\widehat{\boldsymbol{G}}$ is the optimal active set and thus $\left(\widehat{\boldsymbol{G}}^{-1}\mathbf{1}\right)_i > 0$ for all i in $\widehat{\boldsymbol{G}}$. The balanced neighborhood equity condition implies that

$$\lambda = \frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}.$$

At any solution which satisfies the balanced equity condition and allocates a fraction s of shares to agents, the team performance

$$Y^* = \mathbf{1}^T\boldsymbol{a}^* + \frac{\beta}{2}(\boldsymbol{a}^*)^T\widehat{\boldsymbol{G}}\boldsymbol{a}^*$$

can be rewritten as

$$Y^* = \left(\frac{P'(Y^*)}{1 - \beta P'(Y^*)\lambda} + \frac{\beta P'(Y^*)^2\lambda}{2(1 - \beta P'(Y^*)\lambda)^2}\right)s. \tag{22}$$

Applying (42) to the linear output setting with complementarity parameter β, we can write the profit for the principal under this contract as

$$V(s,\beta) = \kappa^2 s(1-s)\left(\frac{1}{1-\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}} + \frac{\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}}{2\left(1-\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}\right)^2}\right).$$

So for a fixed β for which $\boldsymbol{\tau}^*$ is an optimal contract, the total payments under this contract solves the optimization problem

$$V^*(\beta) = \max_{s\in[0,1]} s(1-s)\left(\frac{1}{1-\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}} + \frac{\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}}{2\left(1-\beta\kappa\frac{s}{\mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}}\right)^2}\right).$$

We will characterize the solution to this optimization problem. We define $k^* := \mathbf{1}^T\widehat{\boldsymbol{G}}^{-1}\mathbf{1}$ and claim that $\beta\kappa < k^*$. We must have $\beta \in \left(0, \frac{1}{\kappa}\frac{1}{\rho(\boldsymbol{T}\widehat{\boldsymbol{G}})}\right)$ by our assumption that equilibrium team performance is in $[0,\overline{Y}]$. Observe that for any fixed $s\in[0,1]$, the balance constant $\lambda = s/k^*$ is an eigenvalue for the matrix $\boldsymbol{T}\widehat{\boldsymbol{G}}$ with right eigenvector $\boldsymbol{\tau}$. Thus we have

$$\frac{s}{k^*} \le \rho(\boldsymbol{T}\widehat{\boldsymbol{G}}) < \frac{1}{\beta\kappa}.$$

Choosing $s=1$ then verifies the claim $\beta\kappa < k^*$.

We now return to the problem of maximizing $V(s,\beta)$. Taking the partial derivative with respect to s, we find

$$\frac{\partial V(s,\beta)}{\partial s} = \frac{k^*\kappa^2\left(-(\beta\kappa)^2 s^3 + 3\beta\kappa k^* s^2 - 4(k^*)^2 s + 2(k^*)^2\right)}{2\left(k^*-\beta\kappa s\right)^3}.$$

It suffices to study the behavior of $V(s,\beta)$ when $s\in[0,1]$. Define

$$p(s,\beta) := -(\beta\kappa)^2 s^3 + 3\beta\kappa k^* s^2 - 4(k^*)^2 s + 2(k^*)^2$$

to be the numerator of $V(s,\beta)$. The partial derivative of $p(s,\beta)$ with respect to s is

$$\frac{\partial p(s,\beta)}{\partial s} = -3(\beta\kappa)^2 s^2 + 6\beta\kappa k^* s - 4(k^*)^2 < -3(\beta\kappa s + k^*)^2.$$

Since the right-hand expression is strictly negative, the function $p(s,\beta)$ is strictly decreasing in $s\in[0,1]$. Thus $p(\cdot,\beta)$ has only one real root for each β.

We claim that this root lies in $\left(\frac{1}{2},1\right)$. At $s=\frac{1}{2}$, we have

$$p\left(\frac{1}{2},\beta\right)=\left(-(\beta\kappa)^2\cdot\frac{1}{8}+3\kappa\beta k^*\cdot\frac{1}{4}\right)>0,$$

for any $\beta\kappa<k^*$. At $s=1$, we have

$$p(1,\beta)=(k^*-\beta\kappa)(\beta\kappa-2k^*)<0,$$

for any $\beta\kappa<k^*$. This proves the claim.

For $s\in[0,1]$, the denominator of $V(s,\beta)$ is strictly positive for any $\beta\kappa<k^*$. So for each β, the sum of shares s at the optimal contract is characterized by $p(s,\beta)=0$. We calculate

$$\frac{\partial p(s,\beta)}{\partial\beta}=3\kappa k^*s^2-2\beta\kappa^2s^3=\kappa s^2\left(3k^*-2\beta\kappa s\right)>0,$$

where the inequality holds for all $s\in(0,1)$. Since $p(s,\beta)$ is strictly decreasing in s for each β, the sum of shares s at the optimal contract is increasing in β.

A.12. **Details of Example from 5.4.** We verify the fact from the example by explicitly solving for the marginal contribution and centrality. For a contract $\boldsymbol{\tau}$, equilibrium actions are

$$a_1^*=\frac{P'(Y^*)}{1-P'(Y^*)^2\tau_1\tau_2}\left((1+\delta)\tau_1+P'(Y^*)\tau_1\tau_2\right),$$
$$\text{and}\quad a_2^*=\frac{P'(Y^*)}{1-P'(Y^*)^2\tau_1\tau_2}\left(\tau_2+(1+\delta)P'(Y^*)\tau_1\tau_2\right).$$

Since both agents are active, we must have $P'(Y^*)^2\tau_1\tau_2<1$.

The marginal contributions of the agents are

$$\alpha_1=1+\delta+a_2^*\quad\text{and}\quad\alpha_2=1+a_1^*.$$

The difference in their marginal contributions is

$$\alpha_1-\alpha_2=\delta+\left(a_2^*-a_1^*\right),\tag{23}$$

$$=\delta+\frac{P'(Y^*)}{1-P'(Y^*)^2\tau_1\tau_2}\left(\tau_2-\tau_1(1+\delta)+\delta P'(Y^*)\tau_1\tau_2\right),\tag{24}$$

$$=\frac{\delta+P'(Y^*)\tau_2-P'(Y^*)\tau_1(1+\delta)}{1-P'(Y^*)^2\tau_1\tau_2}.\tag{25}$$

The equation above will be used to show that $\alpha_1 > \alpha_2$ at an optimal contract.

The centrality of each agent is

$$\boldsymbol{c}^\top = \boldsymbol{\alpha}^\top \left[\boldsymbol{I} - P'(Y^*)\boldsymbol{T}\boldsymbol{G}\right]^{-1}.$$

This simplifies to

$$c_1 = \frac{\alpha_1 + P'(Y^*)\tau_2\alpha_2}{1-\tau_1\tau_2 P'(Y^*)^2} \quad \text{and} \quad c_2 = \frac{\alpha_2 + P'(Y^*)\tau_1\alpha_1}{1-\tau_1\tau_2 P'(Y^*)^2}. \tag{26}$$

At the optimal contract, the balance condition $\alpha_1 c_1 = \alpha_2 c_2$ must hold. Substituting (26) to the balance condition, we obtain

$$\alpha_1^2 - \alpha_2^2 = P'(Y^*)\alpha_1\alpha_2\left(\tau_1 - \tau_2\right). \tag{27}$$

We want to show that $\tau_1 > \tau_2$, and will do so by ruling out $\tau_1 = \tau_2$ and $\tau_1 < \tau_2$. Suppose first that $\tau_1 = \tau_2$. Substituting this contract in (23), we get

$$\begin{aligned}\alpha_1 - \alpha_2 &= \frac{\delta - \delta P'(Y^*)\tau_1}{1 - P'(Y^*)^2\tau_1^2}, \\ &= \frac{\delta}{1 + P'(Y^*)\tau_1}, \\ &> 0.\end{aligned}$$

This is a contradiction to (27), so $\tau_1 \neq \tau_2$.

Now suppose that $\tau_2 > \tau_1$. Recall that $P'(Y^*)^2\tau_1\tau_2 < 1$. Since $\tau_2 > \tau_1$ by assumption, we get $P'(Y^*)\tau_1 < 1$. Substituting this in (23), we get

$$\begin{aligned}\alpha_1 - \alpha_2 &= \frac{\delta - P'(Y^*)\tau_1\delta + P'(Y^*)\tau_2 - P'(Y^*)\tau_1}{1 - P'(Y^*)^2\tau_1\tau_2}, \\ &> 0.\end{aligned}$$

This is again a contradiction to (27), so we also cannot have $\tau_1 > \tau_2$. Thus, at the optimal contract, we have $\tau_1 > \tau_2$ and consequently $\alpha_1 > \alpha_2$.

A.13. **Proof of Proposition 9.** We begin with a lemma, which adapts Lemma 1.

Lemma 6. *Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract with corresponding equilibrium actions $\boldsymbol{a}^*$ and team performance Y^*. For any agent i, the derivative of team performance* *in* σ_i, *evaluated at* $\boldsymbol{\sigma}^*$, *is*

$$\frac{dY}{d\sigma_i} = l\alpha_i c_i \sum_{s\in\mathcal{S}} P_s'(Y^*) v_s u_i'\left(\sigma_i^* v_s\right),$$

where l *is independent of* i *and* s.

Proof. The steps taken in this proof are exactly the same as those taken in the proof of Lemma 1. We analyze the change in team performance as the equity transfer to an agent is perturbed. Consider an equity payment scheme $\boldsymbol{\tau}^*$ and any agent i. Consider marginally increasing σ_i. The change induced by this perturbation is

$$\frac{\partial Y}{\partial \sigma_i} = \nabla Y(\boldsymbol{a}^*)^T \cdot \frac{\partial \boldsymbol{a}^*}{\partial \sigma_i}, \tag{28}$$

where $\boldsymbol{a}^*$ is the equilibrium action profile for the contract $\boldsymbol{\tau}$. The substance of the proof is analyzing the second term on the right-hand side of (28).

As in Lemma 1, it is without loss to analyze the change in the action of agent i and actions of agents j that take strictly positive actions in profile $\boldsymbol{a}^*$. The analysis from here on focuses on such agents, overloading notation to represent the actions of these agents by $\boldsymbol{a}^*$.

We will show that the change in equilibrium actions $\boldsymbol{a}^*$ as the equity σ_i increases is

$$\frac{\partial \boldsymbol{a}^*}{\partial \sigma_i} = \boldsymbol{H}^{-\frac{1}{2}}\left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1}\boldsymbol{H}^{-\frac{1}{2}}\begin{bmatrix}\boldsymbol{0}\\ \frac{\partial Y}{\partial a_i}\sum_{s\in\mathcal{S}} P_s'(Y)v_s u_i'(\sigma_i v_s)\\ \boldsymbol{0}\end{bmatrix} + \frac{\partial Y}{\partial \sigma_i}\left[\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}\right]^{-1}\boldsymbol{d}. \tag{29}$$

Consider the equilibrium action profile $\boldsymbol{a}^*$. For an agent j, the first-order conditions imply a_j^* must solve the equation

$$C_j'(a_j) = \left(\sum_{s\in\mathcal{S}} P_s'(Y) u_j(\sigma_j v_s)\right)\frac{\partial Y}{\partial a_j}. \tag{30}$$

To arrive at (29), let us implicitly differentiate (30) with respect to σ_i. For all $j \neq i$,

$$C_j''(a_j^*)\frac{\partial a_j^*}{\partial \sigma_i} = \left(\sum_{s\in\mathcal{S}} P_s'(Y) u_j\left(\sigma_j v_s\right)\right)\left(\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial a_k \partial a_j}\cdot\frac{\partial a_k^*}{\partial \sigma_i}\right) \tag{31}$$

$$+\frac{\partial Y}{\partial a_j}\cdot\frac{\partial Y}{\partial \sigma_i}\cdot\sum_{s\in\mathcal{S}}P_s''(Y)u_j(\sigma_j v_s). \tag{32}$$

Similarly for $j=i$,

$$C_j''(a_j^*)\frac{\partial a_j^*}{\partial \sigma_i}=\left(\sum_{s\in\mathcal{S}}P_s'(Y)u_j(\sigma_j v_s)\right)\left(\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial a_k\partial a_j}\cdot\frac{\partial a_k^*}{\partial \sigma_i}\right) \tag{33}$$

$$+\frac{\partial Y}{\partial a_j}\sum_{s\in\mathcal{S}}P_s'(Y)v_s u_j'(\sigma_j v_s)+\frac{\partial Y}{\partial a_j}\cdot\frac{\partial Y}{\partial \sigma_i}\sum_{s\in\mathcal{S}}P_s''(Y)u_j(\sigma_j v_s). \tag{34}$$

We can combine (31) and (33) in vector form:

$$\frac{\partial \boldsymbol{a}^*}{\partial \sigma_i}=[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\begin{bmatrix}\boldsymbol{0}\\ \frac{\partial Y}{\partial a_i}\sum_{s\in\mathcal{S}}P_s'(Y)v_s u_i'(\sigma_i v_s)\\ \boldsymbol{0}\end{bmatrix}+\frac{\partial Y}{\partial \sigma_i}[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}.$$

The expression in (29) follows.

Substituting (29) into (28), the change in team performance as the equity payment σ_i increases is

$$\frac{\partial Y}{\partial \sigma_i}=\nabla Y(\boldsymbol{a}^*)^T\boldsymbol{H}^{-\frac{1}{2}}\left[\boldsymbol{I}-\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1}\boldsymbol{H}^{-\frac{1}{2}}\begin{bmatrix}\boldsymbol{0}\\ \frac{\partial Y}{\partial a_i}\sum_{s\in\mathcal{S}}P_s'(Y)v_s u_i'(\sigma_i v_s)\\ \boldsymbol{0}\end{bmatrix}$$
$$+\frac{\partial Y}{\partial \sigma_i}\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}.$$

Applying the definitions of α_i and c_i, we obtain

$$\frac{\partial Y}{\partial \sigma_i}=\alpha_i c_i\sum_{s\in\mathcal{S}}P_s'(Y)v_s u_i'(\sigma_i v_s)+\frac{\partial Y}{\partial \sigma_i}\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}.$$

Rearranging,

$$\frac{\partial Y}{\partial \sigma_i}=\frac{1}{1-\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}}\cdot\alpha_i c_i\sum_{s\in\mathcal{S}}P_s'(Y)v_s u_i'(\sigma_i v_s).$$

Setting $l=\frac{1}{1-\nabla Y(\boldsymbol{a}^*)^T[\boldsymbol{H}-\boldsymbol{U}\boldsymbol{G}]^{-1}\boldsymbol{d}}$ and observing l does not depend on i, we obtain the desired result. □

Proof of Proposition 9. The expected payoff for the principal under equity payment $\boldsymbol{\sigma}$ and corresponding equilibrium actions $\boldsymbol{a}^*$ is

$$\left(1-\sum_{i\in N}\sigma_i\right)\sum_{s\in\mathcal{S}}v_sP_s(Y(\boldsymbol{a}^*)).$$

Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract inducing equilibrium $\boldsymbol{a}^*(\boldsymbol{\sigma}^*)$ with team performance Y^*. Consider agent i such that $\sigma_i^*>0$. Then the first-order condition for σ_i^* implies that

$$\frac{dY}{d\sigma_i}\cdot\underbrace{\left(1-\sum_{i\in N}\sigma_i^*\right)\sum_{s\in\mathcal{S}}v_sP_s'(Y^*)}_{D}=\sum_{s\in\mathcal{S}}v_sP_s(Y^*).$$

The left-hand side is the benefit from increasing σ_i^* while the right-hand side is the expected additional transfer required. Since each outcome occurs with non-zero probability, the summation labeled D is nonzero.

Substituting Lemma 6 in the above equation, we obtain

$$l\alpha_ic_i\sum_{s\in\mathcal{S}}P_s'(Y^*)v_su_i'(\sigma_i^*v_s)=\frac{\sum_{s\in\mathcal{S}}v_sP_s(Y^*)}{D},$$
$$\iff \alpha_ic_i\sum_{s\in\mathcal{S}}P_s'(Y^*)v_su_i'(\sigma_i^*v_s)=\lambda,$$

where $\lambda=\sum_{s\in\mathcal{S}}v_sP_s(Y^*)/(lD)$. Observing that λ is independent of i and the outcome s, the statement of the result follows. □

Appendix B. Optimal Contracts without Assumption 1

A key assumption behind our our main result is that when we perturb the optimal contract in any direction, the induced equilibrium varies in a differentiable way (Assumption 1). This section shows modified balance conditions can continue to hold when the induced equilibrium varies differentiably as the contract is perturbed in some, but not necessarily all, directions. The underlying idea is that the principal may only have available perturbations that preserve some global constraints along with the first-order conditions.

Consider an optimal contract $\boldsymbol{\tau}^*$ and a corresponding equilibrium $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$ that is stable but not strict. Then the implicit function theorem ensures that as we perturb the contract $\boldsymbol{\tau}^*$ to $\boldsymbol{\tau}$ locally, there is an action profile $\boldsymbol{a}(\boldsymbol{\tau})$ satisfying agents' first-order conditions that is continuously differentiable in $\boldsymbol{\tau}$. Since the equilibrium is no longer strict, however, these action profiles need not be equilibria for all τ: if agent i is indifferent to his equilibrium action $a^*(\boldsymbol{\tau}^*)_i$ and some alternative, then after perturbing the contract the locally optimal action $a(\boldsymbol{\tau})_i$ need no longer be globally optimal. Then there could be perturbations for which the principal would prefer $\boldsymbol{a}(\boldsymbol{\tau})$ to $\boldsymbol{a}^*(\boldsymbol{\tau}^*)$, so our full balance conditions need no longer hold. But if there are perturbations such that $\boldsymbol{a}(\boldsymbol{\tau})$ remains an equil. ibrium, then we can obtain balance conditions in the directions of all such perturbations.

Consider increasing payments to agent i in the direction $\boldsymbol{t}_i$, where $t_i(s)$ specifies the change in payments to agent i in each outcome s. We hold fixed payments to all other agents $j \neq i$.

Definition. *We say **equilibrium is maintained** in direction $\boldsymbol{t}_i$ if there exist a one-parameter family $\boldsymbol{\tau}(x)$ of contracts with $\boldsymbol{\tau}(0) = \boldsymbol{\tau}^*$ such that $\tau_i'(0) = \boldsymbol{t}_i$, $\tau_j'(0) = 0$ for $j \neq i$, and such that $\boldsymbol{a}(\boldsymbol{\tau}(x))$ is an equilibrium for x in some neighborhood of 0.*

When there are directions where equilibrium is maintained, we obtain balance conditions in those directions. The statements of these conditions must be modified since we now change payments under multiple outcomes.

Theorem 2. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract inducing equilibrium $\boldsymbol{a}^*$ with team performance Y^*. For all directions $\boldsymbol{t}_i$ such that equilibrium is maintained and $\tau_i^*(s) > 0$ whenever $t_i(s) > 0$, we have*

$$q\alpha_i c_i \left(\sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) u_i'(\tau_i^*(s)) \right) = \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

for a constant q that is independent of the direction $\boldsymbol{t}_i$ and agent i.

Proof. The proof follows a similar approach to the proof of Theorem 1. Consider any agent i receiving a payment at at least one outcome. Also, consider a direction of payment perturbation $\boldsymbol{t}_i$. As before, we can ignore agents that receive a zero payment at the optimal contract as they continue to take a zero action and not contribute to the change in equilibrium team performance. The change induced by such perturbation $\boldsymbol{t}_i$ is

$$dY(\boldsymbol{t}_i) = \nabla Y(\boldsymbol{a}^*)^T \cdot d\boldsymbol{a}^*(\boldsymbol{t}_i),$$

where $dY(\cdot)$ is the directional derivative of equilibrium team performance and $d\boldsymbol{a}^*(\cdot)$ is the directional derivative of equilibrium actions. We state a lemma which characterizes the change in equilibrium actions for such a perturbation.

Lemma 7. *The change in equilibrium team performance as payments to agent i are perturbed in direction $\boldsymbol{t}_i$ is*

$$dY(\boldsymbol{t}_i) = l\alpha_i c_i \sum_{s\in\mathcal{S}} P'_s(Y^*)t_i(s)u'_i(\tau_i(s)),$$

for some constant l.

Proof. Consider the equilibrium action profile $\boldsymbol{a}^*$. For an agent j, the first-order conditions imply a_j^* must solve the equation

$$C'_j(a_j) = \left(\sum_{s'\in\mathcal{S}} P'_{s'}(Y)u_j(\tau_j(s'))\right)\frac{\partial Y}{\partial a_j}. \tag{35}$$

Let us take the directional derivative of (35) in direction $\boldsymbol{t}_i$. The expression we obtain depends on whether $j = i$. For all $j \neq i$,

$$\begin{aligned} C''_j(a_j)da_j^*(\boldsymbol{t}_i) = &\left(\sum_{s\in\mathcal{S}} P'_s(Y)u_j(\tau_j(s))\right)\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial a_k \partial a_j}da_k^*(\boldsymbol{t}_i) \\ &+ \frac{\partial Y}{\partial a_j}dY(\boldsymbol{t}_i)\sum_{s\in\mathcal{S}} P''_s(Y)u_j(\tau_j(s)). \end{aligned}$$

On the other hand, for $j = i$

$$\begin{aligned} C''_j(a_j)da_j^*(\boldsymbol{t}_i) = &\left(\sum_{s\in\mathcal{S}} P'_s(Y)u_j(\tau_j(s))\right)\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial a_k \partial a_j}da_k^*(\boldsymbol{t}_i) \\ &+ \frac{\partial Y}{\partial a_j}\sum_{s\in\mathcal{S}} P'_s(Y)t_j(s)u'_j(\tau_j(s)) + \frac{\partial Y}{\partial a_j}dY(\boldsymbol{t}_i)\sum_{s\in\mathcal{S}} P''_s(Y)u_j(\tau_j(s)). \end{aligned}$$

We can combine these equations to write the resulting expression in vector form as

$$d\boldsymbol{a}^*(\boldsymbol{t}_i) = [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \begin{bmatrix} 0 \\ \frac{\partial Y}{\partial a_i} \cdot \sum_{s \in \mathcal{S}} P_s'(Y) t_i(s) u_i'(\tau_i(s)) \\ 0 \end{bmatrix} + dY(\boldsymbol{t}_i) \, [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \, \boldsymbol{d},$$

where $\boldsymbol{d}$ is a vector with j^{th} element defined as

$$d_j := \frac{\partial Y}{\partial a_j} \cdot \sum_{s \in \mathcal{S}} P_s''(Y) u_j(\tau_j(s)).$$

The change in equilibrium team performance is

$$dY(\boldsymbol{t}_i) = \nabla Y(\boldsymbol{a}^*)^T \, [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \begin{bmatrix} 0 \\ \frac{\partial Y}{\partial a_i} \cdot \sum_{s \in \mathcal{S}} P_s'(Y) t_i(s) u_i'(\tau_i(s)) \\ 0 \end{bmatrix} \\ + dY(\boldsymbol{t}_i) \nabla Y(\boldsymbol{a}^*)^T \, [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \, \boldsymbol{d}.$$

Applying the definitions of α_i and c_i while rearranging the equation, we obtain

$$\begin{aligned} dY(\boldsymbol{t}_i) &= \frac{1}{1 - \nabla Y(\boldsymbol{a}^*)^T \, [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \, \boldsymbol{d}} \alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) u_i'(\tau_i(s)), \\ &= l \alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) u_i'(\tau_i(s)), \end{aligned}$$

where $l = \frac{1}{1 - \nabla Y(\boldsymbol{a}^*)^T [\boldsymbol{H} - \boldsymbol{U}\boldsymbol{G}]^{-1} \boldsymbol{d}}$. Observing that r does not depend on i, we obtain the desired result. $\square$

To prove the theorem, we utilize this expression for change in team performance to analyze the principal's first-order condition. For any agent i, consider a direction of perturbation $\boldsymbol{t}_i$ such that whenever $t_i(s) > 0$, the optimal contract is such that $\tau_i^*(s) > 0$. Because equilibrium is maintained in direction $\boldsymbol{t}_i$, the following principal first-order condition must hold:

$$\underbrace{dY(\boldsymbol{t}_i) \sum_{s \in \mathcal{S}} \left(v_s - \sum_{i=1}^{n} \tau_i(s) \right) P_s'(Y^*)}_{D} = \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

Substituting the expression in Lemma 7, we get

$$lD\alpha_i c_i \sum_{s\in\mathcal{S}} P_s'(Y^*)t_i(s)u_i'(\tau_i(s)) = \sum_{s\in\mathcal{S}} P_s(Y^*)t_i(s).$$

The expression in the theorem follows by taking $q = lD$. □

Appendix C. Sufficient conditions for positive payments

The balance result in Theorem 1 only applies to agents receiving a positive payment under a given outcome. As discussed in Section 5, not all agents necessarily receive positive payments at the optimal contract. In this section, we provide sufficient conditions on the environment which guarantee every agent receives a positive payment at some outcome.

Assumption 3. *The environment is such that:*

(a) The contract giving payments $\tau_i(s) = 0$ for all i and s is not optimal.
(b) For every agent i, $\lim_{\tau\to 0} u_i'(\tau) = \infty$.
(c) For any pair of agents i and j and any action profile $\boldsymbol{a}$,

$$\frac{\partial^2 Y(\boldsymbol{a})}{\partial a_j \partial a_i} \geq 0.$$

Part (a) ensures that the principal finds it optimal to pay at least one agent in the team. Part (b) is a standard Inada condition for the agent's utility. Part (c) says that agents' actions are complements in the sense that an agent's effort increases team performance more when other agents exert more effort.

Under these assumptions, all agents are paid precisely at the outcomes that are more likely when team performance increases slightly.

Proposition 10. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract with induced team performance Y^*. For any agent i and any outcome s,*

$$\tau_i^*(s) > 0 \quad \textit{if and only if} \quad P_s'(Y^*) > 0.$$

In general, it can be optimal to exclude some agents from the optimal team (by offering them no incentives). The proposition states that when agents are sufficiently risk averse and actions are complementary, it is optimal to include all agents. Moreover, all agents are paid under all outcomes that would become more likely if they increased their effort.

The remainder of this section proves the proposition. We begin by stating a key lemma, which will help show that at any optimal contract all agents have positive centralities.

Lemma 8. *At any optimal contract $\boldsymbol{\tau}^*$, the spillover matrix $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ has spectral radius strictly smaller than* 1.

Proof. We will use ρ to denote the spectral radius of $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$. We will show that the spillover matrix cannot have $\rho \geq 1$ at any optimal contract. To do so, we will construct a perturbation of the contract giving the principal a higher payoff.

By definition (see Section 3.1),

$$\boldsymbol{c}^T \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right] = \boldsymbol{\alpha}^T.$$

It is helpful to recall the definitions of the terms in the spillover matrix. The matrix $\boldsymbol{U}$ is diagonal with entries

$$U_{jj} = \sum_{s\in\mathcal{S}} P_s'(Y^*)u_j(\tau_j^*(s)).$$

We showed in Lemma 4 that any agent receives a positive payment under an optimal contract $\boldsymbol{\tau}^*$ only at outcomes where $P_s'(Y^*) > 0$. An implication of this is each diagonal entry in $\boldsymbol{U}$ is positive. The matrix $\boldsymbol{G}$ is non-negative by Part (c) of Assumption 3. The matrix $\boldsymbol{H}$ is diagonal with entries

$$H_{jj} = C_j''(a_j^*).$$

Since we assume cost functions are strictly convex, these diagonal entries are positive. It follows that $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ is non-negative, so by the Perron-Frobenius theorem this matrix has a right eigenvector $\boldsymbol{p}$ with non-negative real entries and a positive real eigenvalue. Then

$$\boldsymbol{c}^T \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]\boldsymbol{p} = \boldsymbol{\alpha}^T\boldsymbol{p},$$

which can be simplified to

$$(1-\rho)\boldsymbol{c}^T\boldsymbol{p} = \boldsymbol{\alpha}^T\boldsymbol{p}. \tag{36}$$

Now, suppose that $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{UGH}^{-\frac{1}{2}}$ has spectral radius $\rho \geq 1$. By assumption, the team performance $Y(\cdot)$ is strictly increasing in each of its arguments. It follows the right-hand side of (36) is positive. If the spillover matrix has spectral radius equal to 1, the left-hand is 0 while the right-hand is positive. Thus, we cannot have spectral radius 1.

We must show we cannot have a spectral radius $\rho > 1$. Since $1 - \rho$ is negative, this implies $\boldsymbol{c}^T\boldsymbol{p}$ is negative. We will construct a direction $\boldsymbol{t}$ such that when the optimal contract $\boldsymbol{\tau}^*$ is perturbed in direction $\boldsymbol{t}$, the agents' individual incentives moves in direction $-\boldsymbol{p}$. The resulting change in the equilibrium team performance is proportional to $-\boldsymbol{c}^T\boldsymbol{p}$, which is positive. This will contradict the optimality of contract $\boldsymbol{\tau}^*$.

We now state a lemma that characterizes the change in team performance when the contract is perturbed in *some* direction $\boldsymbol{t}$, where $t_i(s)$ specifies the change in payments to agent i in each outcome s. (Payments only to agents receiving a positive payment are perturbed.) The result below generalizes Lemma 1 to any arbitrary direction.

Lemma 9. *The change in equilibrium team performance as payments to agents are perturbed in direction $\boldsymbol{t}$ is*

$$dY(\boldsymbol{t}) = l \sum_i \alpha_i c_i \left(\sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) u_i'(\tau_i^*(s)) \right),$$

for some constant l.

The proof of the result above follows the exact approach as the proof of Lemma 7 so we omit it for brevity. We utilize this expression of change in team performance to analyze the principal's first-order condition. The derivative of the principal's objective is:

$$\underbrace{dY(\boldsymbol{t}) \sum_{s \in \mathcal{S}} \left(v_s - \sum_{i=1}^{n} \tau_i(s) \right) P_s'(Y^*)}_{D} - \sum_i \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s)$$

Substituting the expression in Lemma 9, we get

$$lD \sum_i \alpha_i c_i \sum_{s \in \mathcal{S}} P_s'(Y^*) t_i(s) u_i'(\tau_i(s)) - \sum_i \sum_{s \in \mathcal{S}} P_s(Y^*) t_i(s). \tag{37}$$

Suppose $lD > 0$ at the optimal contract (a similar argument will hold when $lD < 0$.). We will show there exists a direction of perturbation $\boldsymbol{t}$ satisfying the following properties:

- Every element of eigenvector $\boldsymbol{p}$ satisfies

$$p_i = -\alpha_i \sum_{s \in \mathcal{S}} P'_s(Y^*) t_i(s) u'_i(\tau^*_i(s)),$$

- $t_i(s) \leq 0$ for any agent i and outcome s, and
- $t_i(s) < 0$ only if $\tau^*_i(s) > 0$.

For each agent i, choose an outcome s_i where he receives a positive payment. (Recall we have already restricted to the set of agents who receive a positive payment at some outcome, so this is possible.) At such an outcome s_i, the probability $P'_{s_i}(Y^*) > 0$. For outcome s_i, define

$$t_i(s_i) := -\frac{p_i}{\alpha_i P'_{s_i}(Y^*) u'_i(\tau^*_i(s_i))}.$$

We have $t_i(s_i) \leq 0$ because $\alpha_i > 0$ and the entries of $\boldsymbol{p}$ are non-negative. For any other outcomes $s \in \mathcal{S} \setminus s_i$, define $t_i(s) := 0$.

Substituting in (37), the derivative of the principal's objective in direction $\boldsymbol{t}$ is

$$-lD\boldsymbol{c}^T\boldsymbol{p} - \sum_i P_{s_i}(Y^*) t_i(s_i) > 0. \tag{38}$$

Recall, by assumption $lD > 0$. Combining this with $\boldsymbol{c}^T\boldsymbol{p} < 0$ it follows that the first term $-lD\boldsymbol{c}^T\boldsymbol{p} > 0$. The inequality in (38) follows from noting that $t_i(s_i) \leq 0$ for every agent i.

Since $\tau^*_i(s) > 0$ whenever $t_i(s) \neq 0$, a sufficiently small perturbation in direction $\boldsymbol{t}$ is feasible. So $\boldsymbol{\tau}^*$ cannot be optimal, which gives a contradiction. We conclude that at the optimal contract $\boldsymbol{\tau}^*$, the spillover matrix $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ has spectral radius $\rho < 1$. □

To characterize whether an agent receives a positive payment, it is useful to know whether the agent's centrality is positive. We can apply (8) to show that the centrality of each agent receiving a positive payment is strictly positive. To see this, recall that centralities are defined by

$$\boldsymbol{c}^T = \boldsymbol{\alpha}^T \left[\boldsymbol{I} - \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}\right]^{-1}.$$

Since at an optimal contract the spectral radius of $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ is strictly smaller than 1, we can expand the right-hand side as a power series:

$$\boldsymbol{c}^T = \boldsymbol{\alpha}^T \sum_{k=0}^{\infty} \left(\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}} \right)^k .$$

The spillover matrix $\boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$ is non-negative (see the proof of Lemma 8) while each entry of $\boldsymbol{\alpha}$ is strictly positive. We conclude that the centrality c_i of each agent receiving a positive payment is strictly positive.

Unfortunately, this does not allow us to conclude that *all* agents receive a payment. Suppose some agent received payment zero under $\boldsymbol{\tau}^*$. The Inada condition guarantees a small payment to that agent under suitable outcomes would provide a large incentive to work. But whether this incentive helps the principal depends on the sign of that agent's centrality. We will now show that agents that do not receive a payment at the optimal contract have a strictly positive centrality as well.

Our definition of centrality in Section 3.1 focused on agents that receive a payment. We will extend the definition to *all* agents. To do so, we extend various other definitions to allow entries for every agent. Define the matrix $\widetilde{\boldsymbol{H}} \in \mathbb{R}^{n\times n}$ to be diagonal with entries

$$\widetilde{H}_{jj} := C_j''(a_j^*).$$

Define the vector $\widetilde{\boldsymbol{\alpha}}$ by

$$\widetilde{\boldsymbol{\alpha}} := \widetilde{\boldsymbol{H}}^{-\frac{1}{2}} \nabla Y(\boldsymbol{a}^*).$$

Define the matrix $\widetilde{\boldsymbol{U}} \in \mathbb{R}^{n\times n}$ to be diagonal with entries

$$\widetilde{U}_{jj} := \sum_{s\in\mathcal{S}} P_s'(Y^*) u_j(\tau_j^*(s)).$$

For any agent that does not receive positive payments under any outcome, the diagonal entry is 0. Define $\widetilde{\boldsymbol{G}} \in \mathbb{R}^{n\times n}$ to be the transpose of the Hessian matrix, i.e.,

$$\widetilde{G}_{jk} := \frac{\partial^2 Y}{\partial a_k \partial a_j}.$$

Observe that the Hessian $\boldsymbol{G}$ defined for agents that receive a payment is a submatrix of $\widetilde{\boldsymbol{G}}$. We can now define all agents centralities given the optimal contract $\boldsymbol{\tau}^*$ by

$$\widetilde{\boldsymbol{c}}^T := \widetilde{\boldsymbol{\alpha}}^T \left[\boldsymbol{I} - \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{H}}^{-\frac{1}{2}}\right]^{-1}. \tag{39}$$

Lemma 10. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract. For any agent i, the centrality $\widetilde{c}_i > 0$.*

Proof. We first verify for agents that receive a payment that their centrality defined in $\widetilde{\boldsymbol{c}}$ is equal to their centrality as defined in $\boldsymbol{c}$. For an ease of notation, let the spillover matrix on *all* agents $\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{H}}^{-\frac{1}{2}}$ and for those with a payment $\boldsymbol{S} = \boldsymbol{H}^{-\frac{1}{2}}\boldsymbol{U}\boldsymbol{G}\boldsymbol{H}^{-\frac{1}{2}}$.

We will show that $\widetilde{\boldsymbol{S}}$ and $\boldsymbol{S}$ have the same non-zero eigenvalues. Consequently, they have the same spectral radius. Suppose μ is an eigenvalue of $\boldsymbol{S}$ with corresponding eigenvector $\boldsymbol{v}$. Then, μ is also an eigenvalue of $\widetilde{\boldsymbol{S}}$. To see this, suppose (without loss of generality) agents with a payment are labeled $\{1, \ldots, k\}$. Rows $(k+1)$ onwards in $\widetilde{\boldsymbol{S}}$ have all zeros. The matrix $\boldsymbol{S}$ is the top-left $(k \times k)$ dimensional submatrix of $\widetilde{\boldsymbol{S}}$. We can define a n-dimensional vector $\widetilde{\boldsymbol{v}}$ as $\widetilde{v}_i := v_i$ when $i \leq k$ and $\widetilde{v}_i := 0$ when $i > k$. It is straightforward to see $\widetilde{\boldsymbol{v}}$ is an eigenvector of $\widetilde{\boldsymbol{S}}$ with eigenvalue μ. We will now prove the other direction. Suppose $\widetilde{\mu}$ is a non-zero eigenvalue of $\widetilde{\boldsymbol{S}}$ with corresponding eigenvector $\widetilde{\boldsymbol{v}}$. Since rows $(k+1)$ onwards in $\widetilde{\boldsymbol{S}}$ have all zeros, it must be that $\widetilde{v}_i = 0$ for components $i \geq (k+1)$. But this implies $\boldsymbol{v}$, corresponding to the first k components of $\widetilde{\boldsymbol{v}}$, is an eigenvector of $\boldsymbol{S}$ with eigenvalue $\widetilde{\mu}$. Applying Lemma 8 tells us the spectral radius of $\widetilde{\boldsymbol{S}}$ is strictly smaller than 1.

So we have the following power series expansion of (39):

$$\widetilde{\boldsymbol{c}}^T = \widetilde{\boldsymbol{\alpha}}^T \sum_{\ell=0}^{\infty} \widetilde{\boldsymbol{S}}^{\ell}. \tag{40}$$

We can write for any $\ell \geq 1$

$$\widetilde{\boldsymbol{S}}^{\ell} = \begin{bmatrix} \boldsymbol{S}^{\ell} & \boldsymbol{J} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{J}$ is a matrix that does not contribute to the centrality. Substituting in (40) and noting that the first k elements of $\widetilde{\boldsymbol{\alpha}}$ are just the vector $\boldsymbol{\alpha}$, we get $\widetilde{c}_i = c_i$ for agents that receive a payment. The centrality $\widetilde{c}_i$ for an agent without a payment will affect overall team performance when their payment at a particular outcome is perturbed.

All that remains to show is that $\widetilde{c}_i > 0$ for every agent. This followsfrom (40): $\widetilde{\boldsymbol{S}}$ is non-negative and every element of $\widetilde{\boldsymbol{\alpha}}$ is strictly positive. $\square$

We now complete the proof of the proposition.

Proof of Proposition 10. The proof involves analyzing the derivative of the principal's objective with respect to payments made to the agents. Consider any agent i. As shown in the proof of Theorem 1, the derivative of the principal's objective with respect to $\tau_i(s)$ is

$$lD\alpha_i c_i u_i'(\tau_i^*(s))P_s'(Y^*) - P_s(Y^*).$$

Forward direction: Let $\mathcal{S}_i^*$ be the set of outcomes at which an agent i receives a positive payment. Then

$$P_s'(Y^*) > 0 \textit{ for all } s \in \mathcal{S}_i^*.$$

The statement of the forward direction is exactly Lemma 4. Note that the arguments to prove the lemma did not require an Inada condition.

Backward direction: Any agent i receives a strictly positive payment at all outcomes where $P_s'(Y^*) > 0$.

By Part (a) of Assumption 3, at the optimal contract $\boldsymbol{\tau}^*$, there exists some agent i receiving a positive payment at some outcome s. Note that from the *forward direction*, we must have $P_s'(Y^*) > 0$. The principal's first-order condition (see proof of Theorem 1), applied to agent i is

$$lD\alpha_i c_i u_i'(\tau_i^*(s))P_s'(Y^*) - P_s(Y^*) = 0.$$

This implies $lD > 0$, because $c_i > 0$ by Lemma 10. For any other agent j, recall from the proof of Theorem 1 that the derivative of the principal's objective in $\tau_j(s)$, is given by the expression

$$lD\alpha_j c_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*).$$

This expression was stated for any agent receiving a positive payment at some outcome, but also holds for agents receiving a zero payment at all outcomes as long as the perturbation is made at an outcome s at which $P_s'(Y^*) > 0$.[20] It is straightforward to verify that c_j is exactly the term that appears in the centrality vector $\widetilde{\boldsymbol{c}}$ defined in the proof of Lemma 10. By the Inada condition on the marginal utility function, the observation that $lDc_j > 0$ (which holds because $c_j > 0$ as shown in Lemma 10), and the fact that $P_s'(Y^*) > 0$

$$\lim_{\tau_j^*(s)\to 0} lD\alpha_j c_j u_j'(\tau_i^*(s))P_s'(Y^*) - P_s(Y^*) > 0.$$

Thus, we cannot have $\tau_j^*(s) = 0$ at an optimal contract. □

Appendix D. Comparative statics as the network changes

This section provides two additional comparative statics results in the quadratic setting of Section 5. We strengthen a link and ask (1) how the optimal contract changes and (2) how the induced team performance changes.

We first describe how the optimal equity shares vary as the network changes. We write $\frac{\partial}{\partial G_{jk}}$ for the derivative in the weight $G_{jk} = G_{kj}$ of the link between j and k. Recall that given a contract, we write $\widetilde{\boldsymbol{G}}$ for the adjacency matrix restricted to active agents.

Proposition 11. *Suppose that under $\boldsymbol{G}$ there is a unique optimal contract $\boldsymbol{\tau}^*$, with agents i, j, and k all active. The derivative of agent i's optimal share as we vary the*

[20]Recall that Lemma 1 was defined for any agent receiving a positive payment at some outcome. We show that the result also holds for agents receiving zero payments at all outcomes, when the perturbation in payments is made in an outcome where $P_s'(Y^*) > 0$. For any agent i taking action $a_i^* > 0$, the first-order conditions at equilibrium imply that a_i^* solves

$$C_i'(a_i) = \left(\sum_{s'\in\mathcal{S}} P_{s'}'(Y)u_i(\tau_i(s'))\right)\frac{\partial Y}{\partial a_i}.$$

We show the equation must also hold if $a_i^* = 0$. To see this, recall that $a_i^* = 0$ if and only if $\tau_i(s) = 0$ at all outcomes s. Consider the first order-condition when $a_i^* = 0$ and $\tau_i(s) = 0$ for all s:

$$C_i'(0) = \left(\sum_{s'\in\mathcal{S}} P_{s'}'(Y)u_i(0)\right)\frac{\partial Y}{\partial a_i}.$$

The left-hand side is zero because $C'(0) = 0$. The right-hand side is zero since $\sum_{s'\in\mathcal{S}} P_{s'}'(Y) = 0$. So the first-order condition holds in this case as well. It follows that the first-order condition binds when payments are perturbed for such an agent at an outcome where $P_s'(Y^*) > 0$.

weight of the link between j and k is

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -(\widetilde{\boldsymbol{G}}^{-1})_{ik}\tau_j^* - (\widetilde{\boldsymbol{G}}^{-1})_{ij}\tau_k^* + \frac{\partial \lambda}{\partial G_{jk}}\frac{\tau_i^*}{\lambda},$$

where λ is the constant from Proposition 4.

The value λ is the total equity in each neighborhood, which Proposition 4 shows is constant across agents. The proof is based on the matrix calculus expression

$$\frac{\partial \boldsymbol{G}(t)^{-1}}{\partial t} = -\boldsymbol{G}(t)^{-1}\frac{\partial \boldsymbol{G}(t)}{\partial t}\boldsymbol{G}(t)^{-1} \tag{41}$$

for the derivative of the inverse of a matrix. The result provides a fairly explicit expression for the impact of changing a link on equity allocations.

Proof of Proposition 11. Proposition 4 tells us that, for all agents such that $\tau_i^* > 0$, we have

$$\boldsymbol{\tau}^* = \lambda \widetilde{\boldsymbol{G}}^{-1}\mathbf{1}.$$

We will use the matrix calculus expression

$$\frac{\partial \boldsymbol{G}(t)^{-1}}{\partial t} = -\boldsymbol{G}(t)^{-1}\frac{\partial \boldsymbol{G}(t)}{\partial t}\boldsymbol{G}(t)^{-1}.$$

Taking the derivative with respect to G_{jk}, we have that

$$\frac{\partial \boldsymbol{\tau}^*}{\partial G_{jk}} = -\lambda\widetilde{\boldsymbol{G}}^{-1}\frac{\partial \widetilde{\boldsymbol{G}}}{\partial G_{jk}}\widetilde{\boldsymbol{G}}^{-1}\mathbf{1} + \frac{\partial \lambda}{\partial G_{jk}}\widetilde{\boldsymbol{G}}^{-1}\mathbf{1}.$$

Analyzing the i^{th} element in this vector gives

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -\lambda(\widetilde{\boldsymbol{G}}^{-1}\mathbf{1})_j(\widetilde{\boldsymbol{G}}^{-1})_{ik} - \lambda(\widetilde{\boldsymbol{G}}^{-1}\mathbf{1})_k(\widetilde{\boldsymbol{G}}^{-1})_{ij} + \frac{\partial \lambda}{\partial G_{jk}} \cdot (\widetilde{\boldsymbol{G}}^{-1}\mathbf{1})_i.$$

The result follows from $\tau_i^* = \lambda(\widetilde{\boldsymbol{G}}^{-1}\mathbf{1})_i$ and the analogous expressions with indices j and k. □

We next look at how team performance under an optimal contract varies as the network changes. Recall that Y^* denotes the equilibrium team performance under an optimal contract. Then $\frac{\partial Y^*}{\partial G_{ij}}$ is the change in this team performance as the weight on the link between agent i and j increases.

Proposition 12. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract. Then the change in equilibrium team performance as G_{ij} varies can be expressed as*

$$\frac{\partial Y^*}{\partial G_{ij}} = \tau_i^* \tau_j^* h,$$

where h does not depend on the identities of i or j.

The proposition says that the increase in team performance from strengthening a link is precisely proportional to the product of the payments given to the two agents connected by that link. The proof gives an explicit formula for the quantity h, which depends on the model parameters and the contract.

The proposition has implications for a designer who can make small changes in the network of complementarities. If the principal could marginally strengthen some links, she would want to focus on links between pairs of agents with large payments.

Proof of Proposition 12. We want to calculate the derivative of the team performance Y^* under the optimal contract as G_{ij} increases. By the envelope theorem, we can calculate this derivative by holding fixed the contract $\boldsymbol{\tau}^*$. To do so, we calculate the derivative of the equilibrium team performance Y^* for a given contract $\boldsymbol{\tau}$ as G_{ij} increases. We will then substitute $\boldsymbol{\tau} = \boldsymbol{\tau}^*$.

Letting $\boldsymbol{a}^*$ be the equilibrium action profile under the contract $\boldsymbol{\tau}$, we calculate

$$\begin{aligned}
\frac{\partial Y}{\partial G_{ij}} &= \frac{\partial \mathbf{1}^T \boldsymbol{a^*} + \frac{1}{2}(\boldsymbol{a}^*)^T \boldsymbol{G} \boldsymbol{a}^*}{\partial G_{ij}}, \\
&= \mathbf{1}^T \frac{\partial \boldsymbol{a^*}}{\partial G_{ij}} + (\boldsymbol{a}^*)^T \boldsymbol{G} \frac{\partial \boldsymbol{a^*}}{\partial G_{ij}} + \frac{1}{2}(\boldsymbol{a}^*)^T \frac{\partial \boldsymbol{G}}{\partial G_{ij}} \boldsymbol{a}^*, \\
&= \left[\mathbf{1}^T + (\boldsymbol{a}^*)^T \boldsymbol{G}\right] \frac{\partial \boldsymbol{a^*}}{\partial G_{ij}} + a_i^* a_j^*.
\end{aligned}$$

The equilibrium action satisfies $\boldsymbol{a}^* = P'(Y)\boldsymbol{T}\boldsymbol{G}\boldsymbol{a}^* + P'(Y)\boldsymbol{T}\boldsymbol{1}$. Thus, we can write

$$\frac{\partial \boldsymbol{a}^*}{\partial G_{ij}} = P'(Y)\boldsymbol{T}\frac{\partial \boldsymbol{G}}{\partial G_{ij}}\boldsymbol{a}^* + P'(Y)\boldsymbol{T}\boldsymbol{G}\frac{\partial \boldsymbol{a}^*}{\partial G_{ij}} + (\boldsymbol{T}\boldsymbol{G}\boldsymbol{a}^* + \boldsymbol{T}\boldsymbol{1})\frac{\partial P'(Y)}{\partial G_{ij}},$$

$$= \begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} P'(Y) + P'(Y)\boldsymbol{T}\boldsymbol{G}\frac{\partial \boldsymbol{a}^*}{\partial G_{ij}} + (\boldsymbol{T}\boldsymbol{G}\boldsymbol{a}^* + \boldsymbol{T}\boldsymbol{1})\, P''(Y)\frac{\partial Y}{\partial G_{ij}}.$$

where $\boldsymbol{T}\frac{\partial \boldsymbol{G}}{\partial G_{ij}}\boldsymbol{a}^*$ is a vector with the i^{th} element equal to $\tau_i a_j^*$, the j^{th} element equal to $\tau_j a_i^*$ and the rest of the elements equal to zero. Solving for $\frac{\partial \boldsymbol{a}^*}{\partial G_{ij}}$ gives

$$\frac{\partial \boldsymbol{a}^*}{\partial G_{ij}} = [\boldsymbol{I} - P'(Y)\boldsymbol{T}\boldsymbol{G}]^{-1}\left(P'(Y)\begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} + \boldsymbol{T}\left[\boldsymbol{1} + \boldsymbol{G}\boldsymbol{a}^*\right] P''(Y)\frac{\partial Y}{\partial G_{ij}} \right).$$

Substituting into the expression for $\frac{\partial Y}{\partial G_{ij}}$ gives

$$\frac{\partial Y}{\partial G_{ij}}\left[1 - (\boldsymbol{1} + \boldsymbol{G}\boldsymbol{a}^*)^T [\boldsymbol{I} - P'(Y)\boldsymbol{T}\boldsymbol{G}]^{-1}\boldsymbol{T}(\boldsymbol{1} + \boldsymbol{G}\boldsymbol{a}^*)\, P''(Y)\right]$$

$$= P'(Y)\left[\boldsymbol{1}^T + (\boldsymbol{a}^*)^T\boldsymbol{G}\right][I - P'(Y)\boldsymbol{T}\boldsymbol{G}]^{-1}\begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} + a_i^* a_j^*.$$

We now use the optimality of $\boldsymbol{\tau}$, which implies the equality $\boldsymbol{a}^* = \boldsymbol{\tau}^*\frac{P'(Y)}{1-\lambda P'(Y)}$ by Proposition 4. Applying this, we obtain

$$\frac{\partial Y^*}{\partial G_{ij}} = \tau_i^*\tau_j^* P'(Y^*)^2 \frac{\left(\frac{2}{(1-\lambda P'(Y^*))^3} + \frac{1}{(1-\lambda P'(Y^*))^2}\right)}{1 - \frac{P''(Y^*)\sum_i \tau_i^*}{(1-\lambda P'(Y^*))^3}}.$$

The right-hand side has the desired form. ∎

Appendix E. Proofs for Characterizing the Active Set

Proof of Lemma 3. By Proposition 4, there exists a constant c such that for all agents that get a strictly positive equity at the optimal solution, $(\boldsymbol{G\tau}^*)_i = \lambda$. At any solution which satisfies the balanced equity condition and allocates a fraction s of shares to agents, the team performance $Y^* = \mathbf{1}^T \boldsymbol{a}^* + \frac{1}{2}(\boldsymbol{a}^*)^T \boldsymbol{G}\boldsymbol{a}^*$ can be rewritten as

$$Y^* = \left(\frac{P'(Y^*)}{1 - P'(Y^*)\lambda} + \frac{P'(Y^*)^2 \lambda}{2(1 - P'(Y^*)\lambda)^2} \right) s. \tag{42}$$

We will conclude from the above expression that team performance is increasing in λ. For a given c, the team performance $Y^*(\lambda)$ is the solution to $f(y, \lambda) = y$, where we define

$$f(y, \lambda) := \left(\frac{P'(y)}{1 - P'(y)\lambda} + \frac{P'(y)^2 \lambda}{2(1 - P'(y)\lambda)^2} \right) s.$$

By assumption $P(\cdot)$ is concave and twice differentiable, so $f(y, \lambda)$ is decreasing in y. Since we have also assumed $P(\cdot)$ is strictly increasing, we have $\frac{\partial f}{\partial \lambda} > 0$ for all y and thus $Y^*(\lambda)$ is increasing in λ. $\square$

Proof of Proposition 7. We consider the success probability objective as the argument is essentially the same for both objectives. By Lemma 3, any optimal allocation maximizes $(\boldsymbol{G\tau})_i$ for active agents i among allocations $\boldsymbol{\tau}$ satisfying the balanced equity condition.

Let $\overline{g} = \max_{i,j} G_{ij}$ and choose i and j such that the link between i and j obtains this maximum weight. Setting $\tau_i = \tau_j = \frac{1}{2}$ and all other $\tau_k = 0$ gives $(\boldsymbol{G\tau})_i = (\boldsymbol{G\tau})_j = \overline{g}/2$. We now show this value cannot be obtained with an active set with diameter greater than 2.

Suppose there is an optimal allocation $\boldsymbol{\tau}^*$ with an active set A^* with diameter greater than 2. Choose active agents i and j such that the distance between i and j is at least 2. The subsets $\{i\}, \{j\}, N(i) \cap A^*$, and $N(j) \cap A^*$ of the active set are all disjoint.

The balanced equity condition implies that $(\boldsymbol{G\tau}^*)_i = (\boldsymbol{G\tau}^*)_j = \lambda$ for some constant λ, and we have

$$\begin{aligned} 2\lambda &= (\boldsymbol{G\tau}^*)_i + (\boldsymbol{G\tau}^*)_j \\ &\leq \overline{g} \sum_{k \in N(i) \cup N(j)} \tau_k^* \end{aligned}$$

$$< \overline{g},$$

where the last inequality holds because $\tau_i^* > 0$ so $\sum_{k \in N(i) \cup N(j)} \tau_k^* < 1$. Since we showed we can obtain a value of $\lambda = \overline{g}/2$, this contradicts the optimality of $\boldsymbol{\tau}^*$. $\square$

Proof of Proposition 8. We consider the success probability objective as the argument is essentially the same for both objectives. By Lemma 3, any optimal allocation maximizes the constant $\lambda = (\boldsymbol{G}\boldsymbol{\tau})_i$ for active agents i among allocations $\boldsymbol{\tau}$ satisfying the balanced equity condition.

Let the size of the maximum clique in the network be $\overline{k}$. Proposition 4 implies that the optimal allocation with active set a clique of size k gives all active agents equal shares $\frac{1}{k}$. Under this allocation, the balanced equity condition holds with constant $\lambda = \frac{\overline{k}-1}{\overline{k}}$.

Suppose an allocation $\boldsymbol{\tau}$ satisfies the balanced neighborhood equity condition with constant $\lambda > \frac{\overline{k}-1}{\overline{k}}$. We will show the active set under this allocation must contain a clique of size at least $\overline{k}+1$, which contradicts our assumption that the size of the maximum clique is $\overline{k}$.

Define

$$k^* = \arg\max_{k \in \mathbb{Z}} \left\{ \lambda - \left(\frac{k-1}{k} \right) > 0 \right\}.$$

We will show that the active set under $\boldsymbol{\tau}$ contains a clique of size $k^* + 1 > \overline{k}$. Call the set of vertices of this active set by A^*. First observe that the equity that each agent gets is at most $(1-\lambda)$. This is because each agent's neighbors receive equity shares summing to λ and the total of all equity shares is 1.

We will define a sequence of agents $i^0, \ldots, i^{k^*}$ inductively such that $i^0, i^1, \ldots, i^k$ is a clique for all k. Fix some i^0 in the active set and define $N_S(i^0) := N(i^0)$. Given $i^0, i^1, \ldots, i^k$ for any $k < k^*$, we define

$$N_S(i^k) := \bigcap_{l=0}^{k} N(i^l).$$

Given $i^0, \ldots, i^k$ with $0 \leq k < k^*$, we want to choose i^{k+1} to be an arbitrary agent in $N_S(i^k)$. To do so, we must show $N_S(i^k)$ is non-empty.

We will prove that the total equity in $N_S(i^k)$ is at least $(k+1)\lambda - k$, i.e.,

$$\sum_{i \in N_S(i^k)} \tau_i \geq (k+1)\lambda - k.$$

We show this by induction on k. The base case $k = 0$ holds by the balanced neighborhood equity condition.

The inductive hypothesis is

$$\sum_{i \in N_S(i^{k-1})} \tau_i \geq k\lambda - (k-1).$$

This implies

$$\sum_{i \in N(i^0) \setminus N_S(i^{k-1})} \tau_i \leq (k-1)(1-\lambda) \tag{43}$$

since $\sum_{i \in N(i^0)} \tau_i = \lambda$.

Since i^k is active, we also have $\sum_{i \in N(i^k)} \tau_i = \lambda$. We can decompose the equity in this neighborhood, potentially along with additional agents' shares, as

$$\sum_{i \in A^* \setminus N(i^0)} \tau_i + \sum_{i \in N(i^0) \setminus N_S(i^{k-1})} \tau_i + \sum_{i \in N_S(i^k)} \tau_i \geq \lambda.$$

By the balanced neighborhood equity condition for agent i^0 and (43), this implies

$$\sum_{i \in N_S(i^k)} \tau_i \geq (k+1)\lambda - k,$$

completing the induction. Since $\lambda > \frac{k^*-1}{k^*}$, this implies that $N_S(i^k)$ is non-empty for each $k \in \{1, \ldots, k^* - 1\}$. So we can construct $i^0, \ldots, i^{k^*}$ as described above.

By construction, the subnetwork $\{i^0, \ldots, i^{k^*}\}$ is a clique of size $k^* + 1$. Since we have assumed the maximum clique has size $\overline{k}$, this contradicts the existence of an allocation $\boldsymbol{\tau}$ satisfying the balanced neighborhood equity condition with constant $\lambda > \frac{\overline{k}-1}{\overline{k}}$. Thus the maximum clique must be an optimal allocation. $\square$