INCENTIVE DESIGN WITH SPILLOVERS

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Abstract. A principal uses bonuses conditioned on stochastic outcomes of a team project to elicit costly effort from the team members. We characterize the allocation of incentive pay across agents and outcomes in any optimal contract by leveraging insights from the theory of network games. Bonuses are proportional to (i) individual productivity and (ii) organizational centrality, defined as being complementary to productive agents. Our results generalize Holmstrom's characterization of optimal single-agent contracts to the multi-agent case.

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1. Introduction

A popular method of motivating the members of a team to work toward a common goal is giving them performance incentives that depend on jointly achieved outcomes. Compensation instruments of this form commonly used in practice include options on the firm's stock, bonuses for achieving sales targets, and profit-sharing. How should such incentive schemes be designed, and how should they take into account the structure of production on the team?

We examine these questions in a simple non-parametric model of a team working on a joint project. Each worker chooses how much costly effort to exert. These actions jointly determine a real-valued *team performance*—for example, the quality of a product. This performance, in turn, determines a probability distribution over possible project *outcomes*—observable realizations that yield monetary revenues for a principal. For example, the outcome may be the sales of the product, with the probability of various sales outcomes depending on the team's performance; the uncertainty reflects stochastic factors outside the team's control. While it is not possible to write contracts contingent on individual actions or the team performance, the principal can commit to a contract specifying payments to each agent contingent on each possible project outcome. The principal's goal is to design this contract in a way that maximizes profit: revenue minus bonus payments.

A principal designing such a contract should take the team's production function into account. To illustrate why, imagine that the principal slightly adjusts the contract of particular agent, Bengt, in a way that motivates him to take a slightly different, higher-cost action. In team production, changing one team member's action can change the (local) effects of other agents' actions on team performance. That, in turn, changes the incentives that these other agents face, even if the payments promised to them in various outcomes do not change. For example, those whose actions are complements to Bengt's are now motivated to work harder, while those whose actions are substitutes have incentives to free-ride on his higher effort. Ultimately, all agents' equilibrium incentives and actions may respond to the initial perturbation, and this makes the design of all the different agents' contracts interconnected.

We do not have a good understanding of how the principal should account for the structure of the team's production function in this design problem. When are there gains available from reallocating incentive pay among agents? At an optimal allocation of incentive pay, how do agents' contracts depend on organizational roles?

Our main contribution in this paper is a characterization, without parametric assumptions, of optimal incentives in the environment we have described. In brief, the results state that optimal contracts must allocate steeper incentives to agents who have higher productivity and those who are organizationally central, in the sense that they have high direct and indirect complementarities with productive agents.

Central to these results is a concise description of the marginal benefit to the principal of allocating incentive pay to an agent in any contract (optimal or not). This marginal benefit is proportional to three terms. The first term, an agent's *marginal contribution*, is the partial derivative of team performance in an individual's action, holding others' actions fixed. The second term is called an agent's *total spillover*. It captures the total equilibrium effect on team performance of increasing the agent's incentives to take higher actions, accounting for all the spillover effects discussed above. The third term is an agent's *marginal utility of money*: it accounts for the fact that an agent who has a low valuation of an additional dollar is, all else equal, a less responsive and less appealing recipient of incentive pay.

Our first main result is that, generically, optimal contracts satisfy a balance condition: the three-term product described above is equal across all agents receiving any performance pay. It turns out that the balance condition is necessary if the principal does not want to shift compensation across agents in any state—something that must hold at any optimum. The condition is interpretable, identifying the quantities that must be measured to assess the optimality of a contract. If the condition is not satisfied, the result also yields guidance for modifying a suboptimal incentive scheme to a better one.

Our model allows quite flexible production functions for the team. For example, the team's production function could depend jointly on the efforts of arbitrary subsets of the team—working in threes, fours, and so on. But our characterization of optimal contracts depends (locally) only on bilateral spillovers. Indeed, a key observation behind our results is that to check for localoptimality of a contract, one can approximate the effort provision game with a quadratic network game. Links in the network record bilateral spillovers, and the relevant centrality terms are essentially Bonacich centralities in this network.

We next apply our balance result to describe how incentives are allocated across agents and across states. Agents' marginal utilities across outcomes obey a proportionality condition: if i's marginal utility is twice j's under one outcome, the same holds under any other outcome. When utilities are identical, for example, agents' compensations are ranked across all outcomes by a single ordering. We also characterize how bonuses should be allocated across different outcomes, showing that under concave utility, agents should be paid more in outcomes that are less likely and more sensitive to changes in output. This generalizes a famous Holmström (1979) result on optimally targeting performance pay across outcomes to multi-agent teams.

In a parametric case where the team's performance is determined by a quadratic network game with complementarities (as in Ballester, Calvó Armengol, and Zenou (2006)), we derive a simple relationship between the optimal payment to an agent and his network position. For this exercise, we assume that agents are risk neutral and differ only in their network positions. These assumptions imply that marginal costs of compensation and productivities are equal across agents, so the optimal contract balances centralities in an endogenous network of spillovers. The resulting incentives are quite different than those that appear in related models, such as payments proportional to agents' centralities (as in Mayol (2023) and Milán and Dávila (2024)). To illustrate this, we show optimal payments and outcomes can respond in counterintuitive ways to the network of complementarities (but this would not happen if payments were proportional to centrality).

We also study settings where contracts are constrained to take specific forms, such as equity pay—a contract linear in the principal's revenue. Even in this more restrictive contracting environment, our results imply that a compensation index computed at an agent's optimal equity share is proportional to the product of his productivity and centrality. This finding shows our insights are applicable in realistic settings where contracts cannot be perfectly tailored to states.

Related literature. Broadly, we contribute to the literature on incentive design when production features spillovers across agents. This is related to the literature on moral hazard in teams, going back to the classic work of Holmström (1982). We adapt that model allowing for a flexible production function and specification of the principal's signal about the team's output, generalizing the "imperfect observability" modeling of Holmström (1979) to the multi-agent case. Our model uncovers a set of simple relationships between optimal compensation and effort spillovers that are new to this literature.[1]

The general topic of optimally setting incentives in the presence of spillovers has recently attracted interest in the literature on networks. This includes, in addition to the work cited above, papers such as Bloch (2016), Galeotti, Golub, and Goyal (2020), Belhaj and Deroïan (2018), and Shi (2022). Most closely related, contemporaneous papers by Mayol (2023) and Milán and Dávila (2024) study optimal contracts in a quadratic network game framework. Their analyses are closest to our applications in Section 5, though we find quite different contracts are optimal.[2] Our main contribution to this literature is a study of a natural and non-parametric formulation, both in terms of the production function and the form of incentives. We show that network game techniques permit some general characterizations of optimal outcomes without the strong parametric assumptions common in the network games literature.

The problem of designing multi-agent contracts has also recently attracted attention in the algorithmic game theory community. Dütting, Ezra, Feldman, and Kesselheim (2023) consider the problem of efficiently computing an optimal *linear* contract

[1]We are also related to papers in this literature on partnerships, such as Legros and Matsushima (1991) and Levin and Tadelis (2005) which analyze optimal sharing of project returns to provide incentives, but which ask questions different from ours.

[2]Agents' compensations under optimal contracts in Mayol (2023) and Milán and Dávila (2024) are variants of their Bonacich centralities. We find that rather than choosing payments equal to Bonacich centralities, the optimal contract equalizes the Bonacich centralities of agents (with respect to an endogenous network of spillovers).

in the multi-agent setting for a specific class of output functions. This matches our analysis of the class of equity contracts in Section 5 and Section 6, but in general we allow arbitrary (potentially non-linear) contracts. The other main contrast between our work and this literature is that the computational literature has focused more on the *extensive margin* question of which agents should be included in a team—i.e., given any incentive to work (Ezra, Feldman, and Schlesinger, 2023, 2024). We address the important complementary question of optimizing on the intensive margin of exactly how much incentive pay to give agents, with or without linearity restrictions on the contract.

Finally, there is a considerable amount of recent pure and applied theoretical work in economics under the general umbrella of contract design for teams. We give just a few examples: Rayo (2007) consider a relational contract setting where soft information about agents' effort levels is observable and used in relational enforcement. Dai and Toikka (2022) study robust multi-agent contracts and give foundations for a principal's use of linear contracts such as equity. Starmans (2022) is motivated by questions related to ours, examining how moral hazard affects the type of team a principal prefers; the modeling approach there is different, with particular additive specifications of effort, in contrast to the flexible technologies we study. Sugaya and Wolitzky (2023) focus on issues of dynamic enforcement in team projects. Our main contribution is a simple static model of optimal allocation of incentives across agents, with obvious potential to interact with the many questions—especially dynamic ones—that are of interest in this literature.

2. Model

There are n agents, $N = \{1, 2, \dots, n\}$, and one principal. The agents take real-valued actions $a_i \geq 0$, which can be interpreted as effort levels. These jointly determine a team *performance*, given by a function $Y : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ which we assume is twice differentiable and strictly increasing in each of its arguments. The team performance determines the project *outcome*, an element of the finite set $\mathcal{S}$. The probability of outcome s is $P_s(Y)$, where for any $s \in \mathcal{S}$, the function $P_s(\cdot)$ is strictly positive and twice differentiable. [3] The principal receives *revenue* v_s from the outcome s.[4]

The principal observes the project outcome but does not observe agents' actions or the team performance Y. (When we use pronouns, we use "she" for the principal and "he" for an agent.) To maximize revenue by incentivizing agents' actions, the principal makes a non-negative payment contingent on the outcome. Upon realization of outcome s, agent i receives payment $\tau_i(s)$. The payments are denoted by $\boldsymbol{\tau} : \mathcal{S} \to \mathbb{R}^n_{\geq 0}$; such a function is called a *contract*.

We consider risk-averse agents and a risk-neutral principal. [5] The utility to agent i from a monetary transfer is given by the function $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is strictly increasing, concave and differentiable . Each agent also has a private cost function $c_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is strictly increasing, strictly convex and twice differentiable in that agent's action. The marginal cost at zero action is zero, that is, $c_i'(0) = 0$. Agent i maximizes the expected payoff from payments minus the private cost of the action a_i,

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} P_s(Y)\, u_i(\tau_i(s)) - c_i(a_i).$$

The payoff for the principal given a contract $\boldsymbol{\tau}$ and team performance Y is the expected payoff of the outcome minus transfers to agents:

$$\sum_{s \in \mathcal{S}} \left(v_s - \sum_i \tau_i(s) \right) P_s(Y).$$

The timing is as follows: The principal commits to a contract $\boldsymbol{\tau}$, following which agents' simultaneously choose actions. Our solution concept for the game among the agents is pure strategy Nash equilibrium, which we refer to as the *equilibrium* for the remainder of the paper.

[3]The assumption that a probability of outcome function is strictly positive is not crucial to the results. It is only made for an easy exposition of statements. In the absence of this assumption, the results must be holding at outcomes that occur with non-zero probability at the optimal team performance.

[4]This should be interpreted as the principal's valuation of that state realizing, gross of any payments she will make to the agents.

[5]The modelling assumption that a principal is risk-neutral is not crucial to the results. The characterization of an optimal contract and its consequences holds for a risk-averse agent as well.

There may be multiple equilibria under some contracts. Given a contract $\boldsymbol{\tau}$, we assume that agents play an equilibrium $\mathbf{a}^*(\boldsymbol{\tau})$ maximizing the principal's expected payoff. Under this selection, a principal's payoff under a contract is well-defined if at least one equilibrium exists. Among such contracts, a contract $\boldsymbol{\tau}$ is *optimal* if no other contract $\widetilde{\boldsymbol{\tau}}$ gives the principal a higher payoff. Implicit in this definition is the assumption that contracts without equilibria can never be optimal.

We illustrate a parametric example of the model. This example will be used throughout the paper to provide some intuition for different results pertaining to an optimal contract, and will be the focus of Section 5.

Example 1. There is a symmetric matrix $\mathbf{G}$, representing an undirected network; so $G_{ij} \geq 0$ is the weight of the link from agent i to j, and $G_{ii} = 0$ for each i. The team performance is the sum of a term that is linear in actions—corresponding to agents' standalone contributions—and a quadratic complementarity term:

$$Y(\mathbf{a}) = \sum_{i \in N} a_i + \frac{\beta}{2} \sum_{i,j \in N} G_{ij} a_i a_j, \text{ for } \beta > 0.$$

There are two possible outcomes $s \in \{0, 1\}$. The revenues from these outcomes are normalized so that $v_1 = 1$ and $v_0 = 0$. These can be interpreted as success or failure of the project. The probability of success is $P(Y)$, where the function $P(\cdot)$ is strictly increasing, concave, and twice differentiable.

Agents are risk-neutral over monetary transfers and have a quadratic cost of effort. Agent i maximizes the expected payoff given by the expression:

$$\mathcal{U}_i = P(Y)\tau_i(1) + (1 - P(Y))\,\tau_i(0) - \frac{a_i^2}{2}.$$

The principal is risk-neutral as well. It will be optimal for the principal to not pay anything to agents at the failure outcome, that is, $\tau_i(0) = 0$ for all agents i (see the proof of equilibrium characterization in Section 5). A contract can then be represented by a n-dimensional vector $\boldsymbol{\tau} \in \mathbb{R}^n_{\geq 0}$.

The example features simple structure in several respects: a quadratic polynomial team performance function, the binary nature of the outcome, and the fact that all agents' efforts are complementary. It may be evident to the reader that the example is an extension of a standard linear-quadratic network game (see Ballester et al. (2006)). The principal's problem is to optimally choose an intervention that affects both private returns to action and the network of complementarities. This is in contrast to existing work on planner interventions — for example Galeotti et al. (2020), Leister, Zenou, and Zhou (2022), Parise and Ozdaglar (2023) — that typically affect a nodes' incentives. The necessary conditions at an optimal contract in the general model leads to a simple characterization of optimal interventions for an exogenous network; a result that may be of independent interest.

2.1. **Remarks on the model.** The team performance Y is real-valued, but the outcome s is discrete. This assumption has little substantive content, since the outcome can be, for example, a revenue rounded to the nearest cent. The fact that outcomes are mediated by a one-dimensional performance level is the main feature of the model, though as we discuss at the end, the key ideas have implications for outcomes determined by a higher-dimensional function of efforts.

The principal is not restricted to a budget of v_s at outcome s. The principal may be willing to lose money at some outcome with the hope of inducing higher action.

We do not assume that equilibria exist under all contracts in the formulation of the model or our analysis. We establish equilibria exist for all contracts in the parametric model in Section 5 and give broader sufficient conditions ensuring equilibria exist under a large class of contracts in Appendix C.

The principal's problem raises several natural questions: Broadly, what do payments at an optimal contract look like? Which agents should receive stronger incentives from the principal; how does this depend on agents' individual characteristics and the spillovers among various agents? Does the principal benefit from paying all agents at an optimal contract? Which outcomes should the principal reward the most?

3. Optimal Contracts

This section states our main result: a balance condition that must hold at optimal contracts. Before giving this result, we define some key notation that will be used in the subsequent analysis.

3.1. **Important notation.** Fix a contract $\boldsymbol{\tau}$ and a corresponding principal-optimal equilibrium $\mathbf{a}^*(\boldsymbol{\tau})$. Let Y^* be the team performance under this equilibrium. We define a series of objects below locally at this equilibrium under this contract, but in many cases we omit the dependence on the equilibrium and the contract to save on notation. Let the *curvature matrix* $\mathbf{H}$ be a diagonal matrix where

$$H_{jj} := c_j''(a_j^*)$$

is the curvature of the cost function for agent j at $\mathbf{a}^*$. Analyzing perturbations as we vary a contract involves analyzing how equilibrium actions vary as incentives change. An agent's best response is less sensitive to increased incentives when $c_j''(a_j^*)$ is larger.

Let $\nabla Y(\mathbf{a}^*)$ be the gradient of $Y(\cdot)$ at $\mathbf{a}^*$, restricted to the agents that take a strictly positive action in $\mathbf{a}^*$. We define the *(normalized) marginal contribution* vector $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} := \mathbf{H}^{-\frac{1}{2}} \nabla Y(\mathbf{a}^*).$$

The i^{th} element κ_i captures the marginal effect of i's action on team performance, rescaled to adjust for the curvature of i's cost function.

Example 1 (*Continued*). Recall that team performance is given by the function

$$Y(\mathbf{a}) = \sum_i a_i + \frac{\beta}{2} \sum_{i,j \in N} G_{ij} a_i a_j,$$

and the cost of effort is quadratic, given by $c_i(a_i) = \frac{a_i^2}{2}$. The formula for κ_i at equilibrium actions $\mathbf{a}^*$ is

$$\kappa_i = 1 + \beta \sum_{j \in N} G_{ij} a_j^*.$$

The marginal effect of i's action on team performance is a standalone effect of 1 in addition to the term $\beta \sum_{j \in N} G_{ij} a_j^*$ arising from complementarities.

To analyze how incentives propagate through the team, we consider the Hessian matrix of the team performance function $Y(\cdot)$ with respect to agent actions. Let $\mathbf{H}$ denote the transpose of the Hessian matrix restricted to agents that take a strictly positive action in $\mathbf{a}^*$. Formally, for agents j and k such that $a_j^* > 0$ and $a_k^* > 0$, define,

$$G_{jk} := \frac{\partial^2 Y}{\partial a_k \partial a_j}.$$

Let the *marginal payment utility* matrix $\mathbf{U}$ be a diagonal matrix where

$$U_{jj} := \sum_{s \in \mathcal{S}} P_s'(Y^*) u_j(\tau_j(s))$$

is the marginal change in agent j's utility when team performance increases.

We next define an object $\bar{\boldsymbol{\kappa}}$ that will capture how a change in an agent's incentives propagates through the team:

$$\bar{\boldsymbol{\kappa}}^T := \boldsymbol{\kappa}^T \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}} \mathbf{U} \mathbf{G} \mathbf{H}^{-\frac{1}{2}}\right]^{-1}. \tag{1}$$

The i^{th} element $\bar{\kappa}_i$ of this vector is the total effect on team performance induced by a marginal change in agent i's incentive to increase a_i. This effect is inclusive of all spillovers on others' incentives through strategic interactions, as we will now illustrate using our running example.

Example 1 (*Continued*). Recall that it is optimal for the principal to not pay any agent upon observing the failure outcome. Thus, a contract can be represented by a n-dimensional vector $\boldsymbol{\tau}$. Let us define the diagonal matrix $\mathbf{T} := \operatorname{diag}(\boldsymbol{\tau})$. The formula for the vector $\bar{\boldsymbol{\kappa}}$ is

$$\bar{\boldsymbol{\kappa}} = \left[\mathbf{I} - \beta P'(Y^*) \mathbf{G} \mathbf{T}\right]^{-1} \boldsymbol{\kappa}.$$

The matrix $\left[\mathbf{I} - \beta P'(Y^*) \mathbf{G} \mathbf{T}\right]^{-1}$ is the Bonacich matrix for network $\mathbf{GT}$. An element of this matrix, indexed by (i, j), captures total discounted paths from agent i to agent j (Ballester et al., 2006). The total effect on team performance due to a unit increase in action from equilibrium by agent i is a weighted sum of these paths across all other agents. The weight on the j^{th} term in the sum is agent j's marginal contribution.

3.2. **Balance condition across agents.** In this section, we present our main result: a balance condition across agents at each outcome realization. We begin by stating a technical condition on the optimal contract $\boldsymbol{\tau}^*$, which holds whenever the equilibrium is stable.

Assumption 1. *A differentiable selection* $\mathbf{a}^*(\boldsymbol{\tau})$ *from the equilibrium correspondence can be defined in a neighborhood of* $\boldsymbol{\tau}^*$.

The above assumption is a regularity condition that permits the use of calculus to study perturbations of the optimal contract.[6]

Theorem 1. *Suppose* $\boldsymbol{\tau}^*$ *is an optimal contract and* Y^* *is the induced team performance. There exists a constant* c_s *such that for any agent* i *receiving a positive payment under* s*, we have*

$$\kappa_i \bar{\kappa}_i u_i'(\tau_i^*(s)) = c_s.$$

This result says that optimal incentives require balance to hold, with the product on the left being equal across agents. Below, we will give more intuition for why this is a necessary condition.

In fact, the proof does not rely on the induced team performance Y^* being optimal. The balance condition at the optimal contract would hold if the principal instead wanted to implement any desired level of performance with minimal (expected) transfers to agents.

The key to the proof is calculating the effect on team performance of increasing an agent's payment under a given outcome. Assumption 1 ensures that these perturbations are well-defined.

[6] If the equilibrium is strict (i.e. no agent is indifferent between his equilibrium action and some other action) at the contract $\boldsymbol{\tau}^*$, this condition is equivalent to the statement that the matrix $\mathbf{I} - \mathbf{H}^{-\frac{1}{2}}(\mathbf{UG} + \mathbf{Q})\mathbf{H}^{-\frac{1}{2}}$ is non-singular, where $\mathbf{Q} := \mathbf{V}\nabla Y(\mathbf{a}^*)\nabla Y(\mathbf{a}^*)^T$, for diagonal matrix $\mathbf{V}$ with entries

$$V_{jj} := \sum_{s \in \mathcal{S}} P_s''(Y^*) u_j(\tau_j(s)).$$

This captures the curvature of the probability of outcome function $P_s(\cdot)$.

Lemma 1. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract with corresponding equilibrium actions $\mathbf{a}^*$ and team performance Y^*. Consider any agent i receiving a positive payment at some outcome. For any outcome s, the derivative of team performance in $\tau_i(s)$, evaluated at $\boldsymbol{\tau}^*$, is*

$$\frac{dY}{d\tau_i(s)} = lP_s'(Y^*)u_i'(\tau_i^*(s))\kappa_i\bar{\kappa}_i,$$

where l is independent of i and s.

A complete proof for the result above is provided in Appendix A. We provide some intuition for the various terms in the expression in the lemma.

Sketch of the proof: The lemma characterizes the effect on team performance of increasing the transfer to agent i under outcome s. We can decompose this effect as the product of three terms:

(i) a term $P_s'(Y^*)u_i'(\tau_i(s))\kappa_i$ capturing the direct effect of increasing $\tau_i(s)$ on i's incentive to exert effort;
(ii) a term $\bar{\kappa}_i$ capturing the spillovers from changing i's incentive to exert effort;
(iii) the constant l, which depends on the curvature of the probability $P_s(Y)$.

We focus on the first two terms and defer treatment of the third term, which is not central in the basic intuition, to the formal proof in the appendix.

The first term measures the principal's ability to directly incentivize agent i by rewarding that agent when outcome s is realized. To calculate this (locally near the initial equilibrium), we compare the curvature of i's utility from taking an action to the curvature of i's cost of an action. The change in agent i's marginal utility of action, as $\tau_i(s)$ increases slightly, is the product of (a) the marginal effect $P_s'(Y^*)$ of team performance on the probability of outcome s, (b) the effect $\frac{\partial Y}{\partial a_i}$ of i's action on team performance, and (c) the marginal utility $u_i'(\tau_i(s))$ of money under outcome s. To obtain agent i's direct response to a stronger incentive, we divide by the curvature $c_j''(a_j^*)$ of the cost (recall $c_j''(a_j^*)$ is the denominator of κ_i).

Multiplying by the second term translates from this direct effect on i's action to the overall change in equilibrium actions. The term $\bar{\kappa}_i$ measures the total spillovers induced by shifting i's incentive to exert effort. Recall the definition

$$\bar{\boldsymbol{\kappa}}^T := \boldsymbol{\kappa}^T \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1}.$$

When $\mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}$ has spectral radius less than one, the expansion

$$\left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1} = \sum_{k=0}^{\infty}(\mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}})^k,$$

gives a helpful intuition. The powers capture the initial increase in i's action, the resulting changes in each agent's best response, the further changes in best responses induced by these, etc. Thus the full summation captures the change in the equilibrium action profile due to the exogenous change in i's incentives—following a standard intuition in network games (Ballester et al., 2006). Finally, the dot product with the marginal contribution vector $\boldsymbol{\kappa}$ translates this change in actions into the change in team performance.

We next discuss some intuition for why Lemma 1 implies Theorem 1. A formal proof is provided in Appendix A. We want to show that the balance condition

$$\kappa_i \bar{\kappa}_i u_i'(\tau_i(s)) = \kappa_j \bar{\kappa}_j u_j'(\tau_j(s)),$$

must hold under an optimal contract. Suppose that the principal would benefit from a slightly higher team performance (the case in which the principal prefers a slightly lower team performance proceeds analogously). Lemma 1 shows that the change in team performance from increasing agent i's payment under outcome s is equal to $\kappa_i \bar{\kappa}_i u_i'(\tau_i(s))$ times terms independent of i, and similarly for agent j. If we had

$$\kappa_i \bar{\kappa}_i u_i'(\tau_i(s)) > \kappa_j \bar{\kappa}_j u_j'(\tau_j(s)),$$

it would be profitable for the principal to pay agent i slightly more and agent j slightly less under outcome s. The same argument holds in the opposite direction, so the balance condition is necessary for the contract to be optimal.

4. Comparisons across agents and outcomes

This section derives consequences of the main result for a comparison of payments made across agents and across outcomes. Section 4.1 shows that agents with symmetric utility functions can be ranked in terms of payments at the optimal contract. Section 4.2 compares the payments a particular agent receives across different outcomes.

4.1. **Ranking agents at the optimal contract.**

Agents can be ranked in terms of payments at the optimal contract. To see this, we establish a relationship between the marginal utilities of agents. An implication of Theorem 1 is that the ratio between any two agents' marginal utilities is the same at every outcome such that both receive positive transfers.

Corollary 1. *Consider an optimal contract $\boldsymbol{\tau}^*$. Let $\mathcal{S}^*_{ij}$ be the set of outcomes at which agents i and j both receive a positive payment. For any outcome $s \in \mathcal{S}^*_{ij}$, we have*

$$\frac{u_i'(\tau_i^*(s))}{u_j'(\tau_j^*(s))} = \frac{\kappa_j \bar{\kappa}_j}{\kappa_i \bar{\kappa}_i}.$$

Intuitively, since outcome probabilities are determined by a joint team performance, agents' incentives should vary across outcomes in similar ways. The corollary formalizes this intuition in terms of marginal utilities in each outcome.[7]

The corollary only applies when agents i and j are both paid at a non-empty set of outcomes. Determining when an agent is paid at a given outcome can be complicated in general, but it is easy to construct settings where the corollary applies. In Appendix B, for example, we give a class of environments in which an Inada condition guarantees that all agents are paid at all outcomes where $P_s'(Y^*) > 0$ (and no other outcomes).

When agents have identical utility functions, agents can be ranked so that an optimal contract provides stronger incentives to more highly ranked agents.

[7] This contrasts with a literature on optimal compensation when the observed outcome can be used to identify individuals who deviated from an desired level of effort (e.g., Holmström (1982) and Legros and Matthews (1993)).

Proposition 1. *Suppose that $\boldsymbol{\tau}^*$ is an optimal contract. If a pair of agents i and j have identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$, then*

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

Th intuition is simple: for two agents that derive the same value from a monetary transfer, the agent with a greater overall effect on team performance at the optimal contract must be receiving a higher payment.

When all agents have an identical utility function, the optimal contract induces a complete ranking on the agents. The relative magnitude of payments across agents depends on the environment. This becomes evident in the parametric example discussed further in Section 5.

4.2. **Payments across outcomes.** A second implication of the main balance result is a relationship between a single agent's marginal utility across outcomes.

Corollary 2. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and Y^* is the induced team performance. If agent i receives positive payments under outcomes s_1 and s_2, then*

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

The corollary states that the marginal utility under each outcome is proportional to the probability of that outcome divided by the marginal change in that probability as team performance increases. That is, agents are paid more in outcomes that are less likely and more responsive to team performance. This result generalizes a result in the single-agent setting of Holmström (1979) concerning how a (single agent's) payments should be allocated across states.

A straightforward application of Corollary 2 characterizes the set of outcomes at which an agent receives a positive payment. If an agent receives a positive payment at some outcome, the outcomes at which it receives a positive payment must all either have a positive marginal probability at equilibrium team performance, or a negative marginal probability. When the team performance function $Y(\cdot)$ is strictly increasing in each of its arguments, the outcomes at which an agent receives a positive payment all have a positive marginal probability at equilibrium team performance. (This is formalized as Lemma 3 in the Appendix).

In the special case that an agent is risk-neutral, a stronger conclusion can be derived on the outcomes at which the agent is paid. Under a mild assumption on the probability of outcome function $P_s(\cdot)$, a risk-neutral agent receives a positive payment in at most one outcome.

Proposition 2. *Suppose that for an optimal contract $\boldsymbol{\tau}^*$ and induced team performance Y^*, there does not exist a pair of outcomes s_1 and s_2 such that*

$$\frac{P_{s_1}(Y^*)}{P'_{s_1}(Y^*)} = \frac{P_{s_2}(Y^*)}{P'_{s_2}(Y^*)}.$$

Then, any risk-neutral agent receives a positive payment in at most one outcome. Moreover, this outcome is unique across all risk-neutral agents.

Risk-averse agents prefer to diversify their payments across outcomes. But a risk-neutral agent does not have this diversification motive, and therefore is best motivated by payment in the outcome that responds most to the team's performance. When all agents are risk-neutral, the optimal contract makes a positive payment to the team at only one outcome. The condition on the functions $P_s(Y)$ holds at the endogenous team performance, but it is straightforward to construct functions $P_s(Y)$ such that the condition does not hold for any possible team performance Y.

5. Applications

Our main result gives necessary conditions for a contract to be optimal. As we have discussed, these conditions involve balancing across agents a product of (i) marginal contributions κ_i; (ii) total effect on team performance $\bar{\kappa}_i$; and (iii) expected marginal utility $u_i'(\cdot)$ evaluated at equilibrium payments.

This result, however, does not directly characterize how agents' incentives and equilibrium actions vary with the environment. Concretely, as an agent's role in the organization changes, the balance condition must remain satisfied at optimal contracts, but this might happen via adjustments in any of the three terms.

In this section, we study a canonical case where these adjustments can be fully characterized—our running example with an exogenously given network of strategic complementarities. Applying our main result to this example yields a complete and explicit characterization of nonnegative optimal payments. Our results on this tie into the literature on network games, because the running example adapts the canonical network game of strategic complements. The crucial distinction is that, rather than taking the agents' incentives to contribute as exogenous, we make them the endogenous outcome of the contract design problem.

Our main message is that optimal incentives and the associated equilibria exhibit some interesting phenomena. In particular, the balance condition is achieved by equalizing agents' endogenous marginal contributions to production, despite their different technological roles. In other words, optimal contracts mute pre-existing centrality differences. This contrasts with the standard results on such games under exogenous incentives. We then flesh some implications, including conflicts of interest that arise between the principal and the agents over technological improvements, which are again caused by optimally designed incentives for effort.

5.1. **Equilibrium characterization.** Throughout Section 5, we study the setup of Example 1 from Section 2. Recall there are two states: 1 (success) and 0 (failure). Because agents' incentives depend only on the difference $\tau_i(1)-\tau_i(0)$ between transfers conditional on success and failure, we can shift payments and assume $\tau_i(0)=0$. This shift can only improve the principal's payoff, so it is without loss of optimality in the principal's problem. Thus, from now on, we will let contracts be described by equity shares τ_i for each agent in the good state, stacked in a vector $\boldsymbol{\tau} \in \mathbb{R}^n_{\geq 0}$.

Proposition 3. *Fixing $\boldsymbol{\tau}$, there exists a unique Nash equilibrium. The equilibrium actions $\mathbf{a}^*$ and team performance Y^* solve the equations*

$$[\mathbf{I} - P'(Y^*)\beta\mathbf{TG}]\mathbf{a}^* = P'(Y^*)\boldsymbol{\tau} \text{ and } Y^* = Y(\mathbf{a}^*), \tag{2}$$

where $\mathbf{T} = \operatorname{diag}(\boldsymbol{\tau})$ is the diagonal matrix with entries $\mathrm{T}_{ii} = \tau_i$.

The characterization is reminiscent of the form of actions in standard network games (Ballester et al., 2006), and applies here despite the nonlinearities due to P.

Note that the result entails a positive equilibrium action for those agents with $\tau_i > 0$, and an action of zero otherwise. An agent is said to be *active* under a given contract $\boldsymbol{\tau}$ if he receives a positive payment $\tau_i > 0$ and *inactive* otherwise. We will focus on characterizing the optimal allocation of shares among active agents. Determining the set of active agents is a complex problem, and interested readers can find results about the active set in an earlier version of this paper (Dasaratha, Golub, and Shah (2023)).

5.2. **Optimal contract.** We now characterize the optimal payments and equilibrium actions among the set of agents receiving positive shares.

Proposition 4. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and $\mathbf{a}^*$ and Y^* are the induced equilibrium actions and team performance, respectively. The following properties are satisfied:*

(a) For any two active agents i and j, we have $\kappa_i = \kappa_j$ and $\bar{\kappa}_i = \bar{\kappa}_j$.
(b) Balanced neighborhood actions: There is a constant $c' > 0$ such that for all active agents i, we have $(\mathbf{G}\mathbf{a}^*)_i = c'$.
(c) Balanced neighborhood equity: There is a constant $c > 0$ such that for all active agents i, we have $(\mathbf{G}\boldsymbol{\tau}^*)_i = c$.

The result states that all active agents make equal marginal contributions and have equal centralities.

The property of balanced neighborhood actions states that for each active agent i, the sum of actions of neighbors of i, weighted by the strength of i's connections to those neighbors in $\mathbf{G}$, is equal to the same number, c'. Similarly, the property of balanced neighborhood equity says that for each active agent i, the sum $\sum_j G_{ij}\tau_j$ of shares given to neighbors of i, weighted by the strength of i's connections to those neighbors in $\mathbf{G}$, is equal to the same number (i.e., is not dependent on i).

Proof of Proposition 4. The characterization of optimal payments in Proposition 4 follows from the balance result derived in Theorem 1. To see this, first observe the following immediate corollary of Theorem 1 in the present environment, which follows from the theorem by observing $u'(\tau_i) = 1$ for all values of τ_i.

Corollary 3. *At an optimal contract* $\boldsymbol{\tau}^*$, *the product* $\kappa_i \bar{\kappa}_i$ *is a constant across all active agents.*

The following lemma is then the key step in proving Proposition 4.

Lemma 2. *If* $\kappa_i \bar{\kappa}_i$ *is constant across all active agents, then,* κ_i *is constant across all active agents.*

The proof of this lemma, which we give in the appendix, starts by differentiating the production function and using the characterization of equilibrium, yielding the formula

$$\begin{aligned}\boldsymbol{\kappa} &= \nabla Y(\mathbf{a}^*),\\ &= \mathbf{1} + \beta \mathbf{G}\mathbf{a}^*,\\ &= \underbrace{[\mathbf{I} - \beta P'(Y^*)\mathbf{G}\mathbf{T}]^{-1}}_{\mathbf{M}} \mathbf{1}.\end{aligned}$$

Corollary 3 implies that the i maximizing κ_i among active agents must minimize $\bar{\kappa}_i$. The fact $\boldsymbol{\kappa} = \mathbf{M}\mathbf{1}$ just derived along with the definition (recall eq. 1) $\bar{\boldsymbol{\kappa}} = \mathbf{M}\boldsymbol{\kappa}$ can be combined to show that this is possible only if $\boldsymbol{\kappa}$ is constant. (In fact the proof of this fact uses only the two equalities just stated and that $\mathbf{M}$ is a nonnegative matrix.)

This conclusion implies part (b) of the proposition using the formula $\boldsymbol{\kappa} = \mathbf{1} + \beta \mathbf{G}\mathbf{a}^*$ found above. To show (c), observe that the definition of $\boldsymbol{\kappa}$ and Lemma 2 imply there is c_1 such that

$$(\mathbf{1}^T[\mathbf{I} - P'(y^*)\beta \mathbf{T}\mathbf{G}]^{-1})_i = c_1$$

for all i. Therefore,

$$1 = c_1 - P'(y^*)\beta c_1 (\mathbf{G}\boldsymbol{\tau})_i$$

for all i, so there exists a constant c such that $(\mathbf{G}\boldsymbol{\tau})_i = c$ for all i (among the subnetwork of active agents). □

5.2.1. *An explicit characterization of the optimal contract.* The system of equations in part (a) of Proposition 4 can be solved explicitly for the optimal payments $\boldsymbol{\tau}^*$ as long as the relevant adjacency matrix $\mathbf{G}$ is invertible, which holds for generic weighted networks. At an optimal solution, the payment to an active agent i is

$$\tau_i^* \propto \left(\widetilde{\mathbf{G}}^{-1}\mathbf{1}\right)_i,$$

where $\widetilde{\mathbf{G}}$ is the subnetwork of active agents for that payment allocation; the same is true for actions, with a different constant of proportionality. This expression captures a sense in which more central agents receive stronger incentives, but $\widetilde{\mathbf{G}}^{-1}\mathbf{1}$ behaves quite differently from standard measures of centrality such as Bonacich centrality. In particular, the inverse $\mathbf{G}^{-1}$ changes non-monotonically as $\mathbf{G}$ changes. This can induce non-monotonicities in the optimal allocation and the resulting actions and utilities. We next describe several comparative statics exercises that highlight some consequences of such non-monotonicities.

5.3. **Comparative statics.** In this section, we explore how the optimal contract, as well as the agents' and principal's payoffs, vary with the technology of production. The simple form of the team performance function Y in our environment, as well as the explicit characterization of incentives and outcomes, facilitate this exercise. We focus on the effects of changes in the network $\mathbf{G}$ and the parameter β describing the strength of complementarities. Section 5.3.1 examines how the optimal team performance depends on the network of complementarities. The results demonstrate some interesting tensions between the principal's and the agents' interests. Section 5.3.2 then explores an interesting practical question about compensation: how the total share of output optimally used for incentive pay depends on the strength of complementarities.

5.3.1. *Varying the network.* We look at how the principal's and agents' payoff vary as the network changes.

Proposition 5. *The principal's payoff is weakly increasing in the edge weight $G_{ij} = G_{ji}$.*

The principal obtains weakly higher profits from an increase in edge weights. However, it need not be the case that agents prefer such a perturbation. We will illustrate

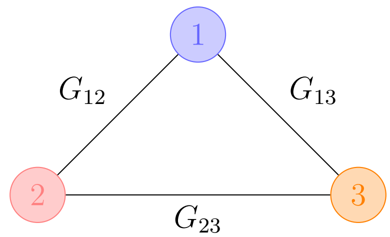


FIGURE 1. Three agent weighted graph with weights G_{12}, G_{13}, and G_{23}.

this through a network on three agents (see Figure 1); general comparative statics can be found in Appendix D.

Without loss of generality, we can assume $G_{12} \geq G_{13} \geq G_{23}$ and choose the normalization $G_{12} = 1$, so that the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & G_{13} \\ 1 & 0 & G_{23} \\ G_{13} & G_{23} & 0 \end{bmatrix}.$$

Figure 2 shows the optimal payments and the corresponding equilibrium payoffs as we vary the link weight G_{23}, under parameter values specified in the caption. Figure 2a depicts optimal payments to each agent as a function of G_{23}. The payment is non-monotonic in own links: increasing G_{23} initially decreases agent 2's payment. The numerical example also illustrates a corresponding non-monotonicity in payoffs: strengthening one of an agent's links can decrease his equilibrium payoff under the optimal contract. Figure 2b depicts the equilibrium payoffs under optimal payments as a function of G_{23}. Strengthening the link between agents 2 and 3 can *decrease* the resulting payoffs for agents 1 and 2.

This finding contrasts with an intuition that one might have from the network games literature, that agents are better off from becoming more central. Under fixed payments, all agents' payoffs are monotone in the network. In the present setting, however, agent 2 can benefit from weakening one of his links.

There is therefore a tension between the network formation incentives of the principal and the agents. Agents may not be willing to form links that would benefit the principal or the team as a whole, even if link formation is not costly.

5.3.2. *Varying complementarities.* We now turn to how total payments change as the complementarity parameter β increases. We study the comparative static in the special case when $P(\cdot)$ is linear in the range of feasible team performance. We assume for simplicity that the optimal allocation is unique, but could easily relax this assumption. The principal faces a trade-off between keeping a larger percentage of its value and using larger payments to encourage workers to exert more effort. The following result states that when complementarities in production are larger, it is optimal to keep a smaller percentage of a larger pie.

Proposition 6. *Suppose that $P(Y) = \alpha Y$ on an interval $[0, \overline{Y}]$ containing the equilibrium team performance under any feasible allocation and that there is a unique optimal allocation $\boldsymbol{\tau}^*$. The sum of agents' payments under the optimal allocation is increasing in β, i.e.,*

$$\frac{\partial \left(\sum_{i \in N} \tau_i^*\right)}{\partial \beta} > 0.$$

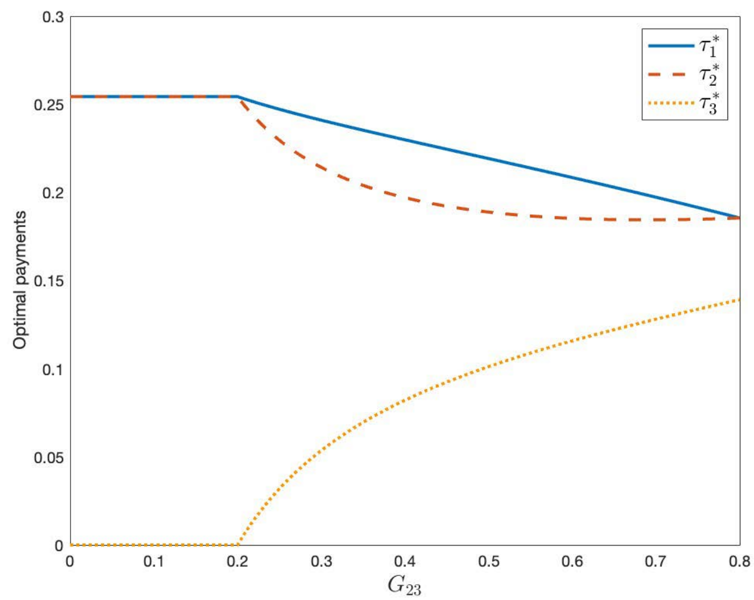
The basic idea behind the proof is that the benefits to retaining more of the firm are linear in the probability of success while the benefits to allocating more shares to workers are convex, and become steeper as complementarities increase.

If $P(Y)$ is strictly concave, there is a trade-off between the concavity of $P(Y)$ and the convexity of $Y(\mathbf{a})$. Depending on which effect is stronger, the total payments made to agents may increase or decrease as complementarities grow stronger.

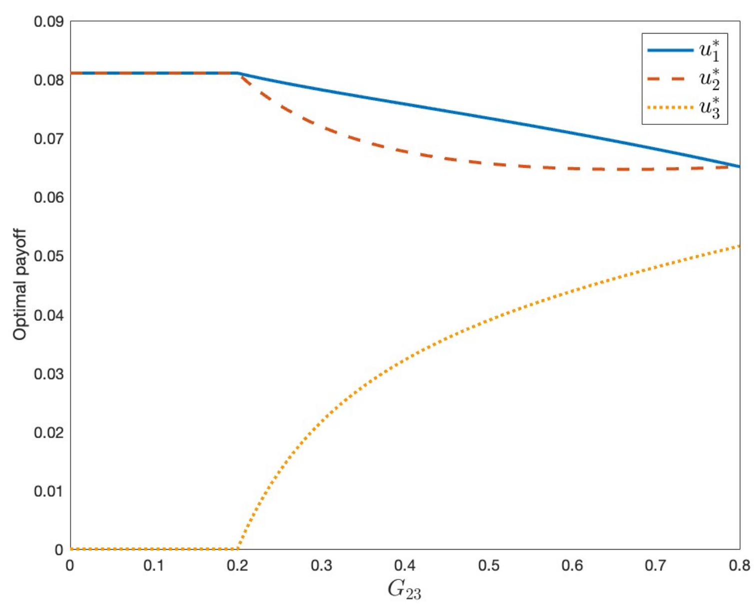
5.4. **Trade-off between marginal contribution and centrality.** In this section, we explore the trade-off between the marginal contribution and spillovers of an agent at the optimal contract. We showed in Section 5.2 that when team performance is

$$Y(\mathbf{a}) = \sum_{i \in N} a_i + \frac{\beta}{2} \sum_{i,j \in N} G_{ij} a_i a_j, \text{ for } \beta > 0,$$

the marginal contribution κ_i is equalized across active agents at the optimal contract. This team performance assume each agent would have the same productivity



(A) Optimal payments



(B) Payoffs under optimal payments

FIGURE 2. The optimal payments and resulting equilibrium payoffs as a function of the weight G_{23}. Here $G_{13} = 0.8$ and $\beta = 0.1$, while $P(Y) = \min\{0.9Y, 1\}$ (the kink is not relevant for the principal's problem). In both diagrams, the curve corresponding to agent 1 is the topmost (solid blue) one; the curve corresponding to agent 2 is the second from the top (dashed red); and the curve corresponding to agent 3 is the lowest (dotted orange) one.

if no other teammates exerted effort, and we now extend the example to relax this assumption.

We illustrate some basic forces through a two agent example. Without loss of generality, the adjacency matrix is

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Suppose team performance is

$$Y(\mathbf{a}) = (1+\delta)a_1 + a_2 + \beta a_1 a_2.$$

We analyze how κ_i varies across agents at the optimal contract for a strictly positive δ. At the optimal contract, the balance condition in Corollary 3 must hold, that is,

$$\kappa_1 \bar{\kappa}_1 = \kappa_2 \bar{\kappa}_2.$$

So the agent with higher individual productivity cannot have larger marginal contribution κ_i and larger spillovers, captured by $\bar{\kappa}_i$.

When $\delta = 1$ and $\beta = 1$, for example, the marginal contributions satisfy $\kappa_1 > \kappa_2$. Applying the balance condition, it must have been that the spillovers are different with $\bar{\kappa}_1 < \bar{\kappa}_2$. Under the principal's favorite contract, the agent with a higher individual productivity has a higher marginal contribution but is less central in the endogenous network of spillovers.

6. Optimal equity pay

The contracts we have described so far are finely tailored to individual outcomes (see Corollary 2). In practice such contract may be difficult to implement, and firms often use simple compensation schemes. Our results can be adapted to characterize optimal contracts within a restricted class. This section provides an illustration by analyzing one widely used incentive scheme: equity pay. Note that equity pay is optimal in our simple running example, but in general the optimal equity contract need not match the optimal unrestricted contract.

An equity pay contract pays each agent a fixed share $\sigma_i v_s$ of the surplus v_s produced by the team. For a given equity contract $\boldsymbol{\sigma}$, the expected payoff to the principal is

$$\left(1 - \sum_{i \in N} \sigma_i\right) \sum_{s \in \mathcal{S}} v_s P_s(Y).$$

The expected payoff to agent i from an equity share σ_i is

$$\mathcal{U}_i = \sum_{s \in \mathcal{S}} u_i\left(\sigma_i v_s\right) P_s(Y) - c_i(a_i).$$

The result below characterizes an optimal equity contract $\boldsymbol{\sigma}^*$. We continue to assume Assumption 1, which now states that there is a neighborhood of $\boldsymbol{\sigma}^*$ in the space of equity contracts where $\boldsymbol{\sigma}(\mathbf{a}^*)$ is continuously differentiable.

Proposition 7. *Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract and Y^* is the induced team performance. There exists a constant c such that for any agent i receiving a positive equity payment, we have*

$$\kappa_i \bar{\kappa}_i \sum_{s \in \mathcal{S}} P_s'(Y^*) v_s u_i'(\sigma_i^* v_s) = c.$$

The proof follows a similar approach to the proof of Theorem 1. It involves analyzing the effect of perturbations to equity payments on the principal's objective. Perturbations in the equity payment of an agent affect payments at all outcomes. The direct effect of increasing σ_i on i's action is proportional to the change in marginal expected utility from payments, which is given by the expression

$$\kappa_i \sum_{s \in \mathcal{S}} P_s'(Y^*) v_s u_i'\left(\sigma_i^* v_s\right).$$

The summation captures the total direct effect of increasing an agent's equity on team performance by aggregating across states, and multiplying by $\bar{\kappa}_i$ includes indirect effects. At an optimal equity contract, the effect of perturbing equity payments on total team performance must be the same for all agents with positive equity.

In general, the balance condition in Proposition 7 characterizing optimal equity contracts does not match the condition in Section 3 characterizing optimal contracts. An optimal contract fine-tunes payments at each outcome, incentivizing agents to exert optimal effort levels. Equity pay imposes a particular relationship between the payments for different outcomes that may be practially convenient but sacrifices some incentive power.

7. Concluding discussion

We have studied a fairly general incentive design problem for a team whose members contribute via unobserved effort. We investigate how optimal contracts depend on the team's production function. Our main contribution is a necessary condition for contract optimality. We show that optimal contracts must satisfy a balance condition across agents receiving positive incentive pay.

The balance result in Theorem 1 generalizes prior work on complementarities in optimal contract design. Beyond the lack of parametric assumptions, our general necessary condition does not require that strategic interactions take any particular form (such as strategic complementarities). What is key to our analysis is studying the perturbations of equilibria, and this is possible with general spillovers. Specific assumptions on spillover structure can, however, be very helpful for guaranteeing equilibrium existence.

There are several natural avenues for future work. First, we have, for simplicity, focused on a one-dimensional function as the mediator between efforts and the contractible outcome. The model could be generalized to a multi-dimensional $Y \in \mathbb{R}^n$. This would introduce new incentive spillovers, but a local analysis of necessary conditions for optimality seems interesting and feasible. Spillovers between different agents and different dimensions of Y will be relevant under the realistic assumption that individual efforts are not perfectly identifiable.

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Appendix A. Omitted proofs

A.1. **Proof of Lemma 1.** We begin by observing that under any optimal contract,

$$a_i^*(\boldsymbol{\tau}^*) = 0 \iff \tau_i^*(s) = 0, \quad \text{for all } s \in \mathcal{S}.$$

That is, at an optimal contract, an agent i exerts zero effort at equilibrium, if and only if it does not receive a payment from the contract at any outcome.[8]

We analyze the change in team performance as the transfers to agents are perturbed. Consider contract $\boldsymbol{\tau}$ and any agent i for which there exists an outcome s' such that $\tau_i(s') > 0$. For any outcome s, consider marginally increasing $\tau_i(s)$. The change induced by this perturbation is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\mathbf{a}^*)^T \cdot \frac{\partial \mathbf{a}^*}{\partial \tau_i(s)}, \tag{3}$$

where $\mathbf{a}^*$ is the equilibrium action profile for the contract $\boldsymbol{\tau}$. The substance of the proof is analyzing the second term on the right-hand side of (3).

First, consider any agent j such that $a_j^*(\boldsymbol{\tau}) = 0$. The change in their equilibrium action due to an increase in $\tau_i(s)$ is zero. The utility to agent j at contract $\boldsymbol{\tau}$ is

$$\mathcal{U}_j = \sum_{s' \in \mathcal{S}} P_{s'}(Y^*) u_j(\tau_j(s')) - c_j(a_j).$$

At contract $\boldsymbol{\tau}$, agent j receives no payment under any outcome, so has a unique best response of $a_j^* = 0$.

It is thus without loss to analyze the change in equilibrium actions of agents j that take a strictly positive action in profile $\mathbf{a}^*$, that is, $a_j^* > 0$. The analysis from here on

[8] Suppose at an optimal contract $\boldsymbol{\tau}^*$, there is an agent i who receives positive payment $\tau_i^*(s) > 0$ under an outcome s but chooses action $a_i^*(\boldsymbol{\tau}^*) = 0$. Then the principal receives a strictly higher payment under the contract $\boldsymbol{\tau}^\dagger$ which sets $\tau_i^\dagger(s) = 0$ and is otherwise equal to $\boldsymbol{\tau}^*$. At this contract, agent i chooses action $a_i = 0$ for any profile of actions $\mathbf{a}_{-i}$ played by other agents. Thus, the equilibrium $\mathbf{a}^*(\boldsymbol{\tau}^*)$ under contract $\boldsymbol{\tau}^*$ is also an equilibrium profile under contract $\boldsymbol{\tau}^\dagger$. Since outcome s occurs with positive probability under any team performance, the principal's expected payments to agents are strictly higher under $\boldsymbol{\tau}^*$ than $\boldsymbol{\tau}^\dagger$. The other direction is straightforward.

focuses on such agents, overloading notation to represent the actions of these agents by $\mathbf{a}^*$.

We will show that the change in equilibrium actions $\mathbf{a}^*$ as the transfer $\tau_i(s)$ increases is

(4)
$$\frac{\partial \mathbf{a}^*}{\partial \tau_i(s)} = \mathbf{H}^{-\frac{1}{2}} \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1} \mathbf{H}^{-\frac{1}{2}} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P_s'(Y) u_i'(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

Consider the equilibrium action profile $\mathbf{a}^*$. For an agent j, the first-order conditions imply a_j^* must solve the equation

$$c_j'(a_j) = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s'))\right) \frac{\partial Y}{\partial a_j}. \tag{5}$$

To arrive at (4), let us implicitly differentiate (5) with respect to $\tau_i(s)$. The resulting expression depends on the identity of agent j in comparison to i, the agent whose payment is perturbed. For all $j \neq i$,

$$\begin{aligned} c_j''(a_j^*) \frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s'))\right) \left(\sum_{k=1}^{n} \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i(s)}\right) + \\ \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i(s)} \cdot \sum_{s' \in \mathcal{S}} P_{s'}''(Y) u_j(\tau_j(s')). \end{aligned} \tag{6}$$

On the other hand, for $j = i$,

$$\begin{aligned} c_j''(a_j^*) \frac{\partial a_j^*}{\partial \tau_i(s)} = \left(\sum_{s' \in \mathcal{S}} P_{s'}'(Y) u_j(\tau_j(s'))\right) \left(\sum_{k=1}^{n} \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \tau_i(s)}\right) \\ + \frac{\partial Y}{\partial a_j} P_s'(Y) u_j'(\tau_j(s)) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \tau_i(s)} \sum_{s' \in \mathcal{S}} P_{s'}''(Y) u_j(\tau_j(s')). \end{aligned} \tag{7}$$

We can combine (6) and (7) to write the resulting expression in vector form below

$$\frac{\partial \mathbf{a}^*}{\partial \tau_i(s)} = \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P_s'(Y) u_i'(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d},$$

where $\mathbf{d}$ is a vector with j^{th} element defined as

$$d_j := \frac{\partial Y}{\partial a_j} \cdot \sum_{s' \in \mathcal{S}} P''_{s'}(Y) u_j(\tau_j(s')).$$

The expression in (4) follows.

Substituting (4) into (3), the change in team performance as the transfer $\tau_i(s)$ increases is

$$\frac{\partial Y}{\partial \tau_i(s)} = \nabla Y(\mathbf{a}^*)^T \mathbf{H}^{-\frac{1}{2}} \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}} \mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1} \mathbf{H}^{-\frac{1}{2}} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} P'_s(Y) u'_i(\tau_i(s)) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \tau_i(s)} \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

Applying the definitions of κ_i and $\bar{\kappa}_i$, we obtain

$$\frac{\partial Y}{\partial \tau_i(s)} = \kappa_i \bar{\kappa}_i P'_s(Y) u'_i(\tau_i(s)) + \frac{\partial Y}{\partial \tau_i(s)} \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

Rearranging,

$$\frac{\partial Y}{\partial \tau_i(s)} = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}} \cdot \kappa_i \bar{\kappa}_i P'_s(Y) u'_i(\tau_i(s)).$$

Setting $l = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}}$ and observing l does not depend on i, we obtain the desired result.

A.2. **Proof of Theorem 1.** The expected payoff for the principal under contract $\boldsymbol{\tau}$ and corresponding equilibrium actions $\mathbf{a}^*$ is

$$\sum_{s' \in \mathcal{S}} \left(v_{s'} - \sum_{i \in N} \tau_i(s') \right) P_{s'}(Y(\mathbf{a}^*)).$$

Suppose $\boldsymbol{\tau}^*$ is an optimal contract inducing equilibrium $\mathbf{a}^*(\boldsymbol{\tau}^*)$ with team performance Y^*. Consider outcome s and any agent i such that $\tau^*_i(s) > 0$. Then the first-order condition for $\tau_i^*(s)$ implies that

$$\frac{dY}{d\tau_i(s)} \underbrace{\sum_{s' \in \mathcal{S}} \left(v_{s'} - \sum_{i \in N} \tau_i^*(s') \right) P'_{s'}(Y^*)}_{D} = P_s(Y^*).$$

The left-hand side is the benefit from increasing $\tau_i^*(s)$ while the right-hand side is the expected additional transfer required. Since $P_s(Y^*) > 0$ by assumption, the summation labeled D is nonzero.

Substituting Lemma 1 in the above equation, we obtain

$$l\kappa_i \bar{\kappa}_i P'_s(Y^*) u'_i(\tau_i^*(s)) = \frac{P_s(Y^*)}{D},$$
$$\iff \kappa_i \bar{\kappa}_i u'_i(\tau_i^*(s)) = c_s,$$

where $c_s = P_s(Y^*)/(lP'_s(Y^*)D)$. Observing that c_s is independent of i, the statement of the result follows.

A.3. **Proof of Corollary 1.** Let $\mathcal{S}_{ij}^*$ have at least 2 outcomes. (If $|\mathcal{S}_{ij}^*| \leq 1$, the statement holds vacuously.) By Theorem 1, for any $s \in \mathcal{S}_{ij}^*$, there is a constant $c_s \neq 0$ such that

$$\kappa_i \bar{\kappa}_i u'_i(\tau_i^*(s)) = c_s, \quad \text{and} \quad \kappa_j \bar{\kappa}_j u'_j(\tau_j^*(s)) = c_s.$$

It follows that

$$\frac{u'_i(\tau_i^*(s))}{u'_j(\tau_j^*(s))} = \frac{\kappa_j \bar{\kappa}_j}{\kappa_i \bar{\kappa}_i}.$$

The right-hand side is independent of s, so the result follows with c_{ij} equal to this right-hand side.

A.4. **Proof of Proposition 1.** We prove a couple of lemmas which help in proving the proposition statement. The first lemma gives a condition which must hold for all outcomes at which an agent receives a positive payment.

Lemma 3. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract and Y^* is the induced team performance. For all s in the set of outcomes $\mathcal{S}_i^*$ where i receives a positive payment,* $P'_s(Y^*) > 0$.

Proof. Consider agent i and let $\mathcal{S}_i^*$ be the set of outcomes at which agent i receives a positive payment. If $\mathcal{S}_i^*$ is the empty set, the result holds vacuously. Otherwise, either

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*, \quad \text{or,} \quad P_s'(Y^*) < 0, \text{ for all } s \in \mathcal{S}_i^*. \tag{8}$$

Recall from the proof of Theorem 1 that

$$l\kappa_i\bar{\kappa}_i P_s'(Y^*) u_i'(\tau_i^*(s)) = \frac{P_s(Y^*)}{\sum_{s'\in\mathcal{S}} \left(v_{s'} - \sum_{i\in N} \tau_i(s')\right) P_{s'}'(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

Since the utility function $u_i(\cdot)$ is strictly increasing, we must have either

$$P_s'(Y^*) > 0 \text{ for } s \in \{s_1, s_2\}, \quad \text{or,} \quad P_s'(Y^*) < 0 \text{ for } s \in \{s_1, s_2\}.$$

The statement in (8) follows. We now show that

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*.$$

The equilibrium condition for agent i is

$$c_i'(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s\in\mathcal{S}} P_s'(Y^*) u_i(\tau_i^*(s)).$$

Since $\sum_{s\in\mathcal{S}} P_s'(Y) = 0$, the equilibrium condition can be rewritten as

$$c_i'(a_i^*) = \frac{\partial Y}{\partial a_i} \sum_{s\in\mathcal{S}_i^*} P_s'(Y^*) \left(u_i(\tau_i^*(s)) - u_i(0)\right).$$

By assumption we have a positive marginal contribution, that is, $\frac{\partial Y}{\partial a_i} > 0$. The cost of effort is strictly increasing, that is, $c_i'(\cdot) > 0$. The utility function $u_i(\cdot)$ is strictly increasing in payments. Thus,

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*$$

as desired. $\square$

The second lemma shows the existence of a common outcome at which agents receiving a positive payment are paid.

Lemma 4. *Suppose* $\boldsymbol{\tau}^*$ *is an optimal contract. Consider a pair of agents* i *and* j*, each with strictly concave utility functions. If there exist outcomes* s_i *and* s_j *such that* $\tau_i^*(s_i) > 0$ *and* $\tau_j^*(s_j) > 0$*, then there exists an outcome* $s \in \mathcal{S}$ *such that*

$$\tau_i^*(s) > 0 \text{ and } \tau_j^*(s) > 0.$$

Proof. Suppose there does not exist an outcome at which both agents receive a positive payment. Thus, the payments $\tau_i^*(s_j) = 0$ and $\tau_j^*(s_i) = 0$. The KKT first-order conditions at optimal contract $\boldsymbol{\tau}^*$ are

$$lD\kappa_k\bar{\kappa}_k u_k'(\tau_k^*(s_k))P_{s_k}'(Y^*) - P_{s_k}(Y^*) = 0 \quad \text{for } k \in \{i, j\}. \tag{9}$$

In addition to the above set of equations, we also have

$$lD\kappa_k\bar{\kappa}_k u_k'(0)P_{s_{\{i,j\}\setminus k}}'(Y^*) - P_{s_{\{i,j\}\setminus k}}(Y^*) \le 0 \quad \text{for } k \in \{i, j\}. \tag{10}$$

Recall $\mathcal{S}_i^*$ is the set of outcomes where agent i receives a positive payment under contract $\boldsymbol{\tau}^*$. Since $P_s'(Y^*) > 0$ for any $s \in \mathcal{S}_i^*$ (see Lemma 3), we must have $lD\kappa_i\bar{\kappa}_i > 0$. Consider the following chain of inequalities for agent i:

$$\frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)} < lD\kappa_i\bar{\kappa}_i u_i'(0) \le \frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)}. \tag{11}$$

Both the inequalities follow from applying (9) and (10) to agent i. We utilize the fact that $u_i(\cdot)$ is strictly concave. We also utilize the observation that, since agent i and j receive a positive payment at outcome s_i and s_j, Lemma 3 tells us that $P_{s_i}'(Y^*) > 0$ and $P_{s_j}'(Y^*) > 0$. Following the same computation for agent j, we obtain the inequalities

$$\frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)} < lD\kappa_j\bar{\kappa}_j u_j'(0) \le \frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)}. \tag{12}$$

This contradicts inequality (11). Thus, if two agents receive a positive payment at some (potentially different) outcomes under the optimal contract, then there must exist an outcome at which both agents receive a positive payment. □

Proof of Proposition 1. Consider agents i and j with identical strictly concave utility functions $u_i(\cdot) = u_j(\cdot)$. The statement trivially holds if either agent i or agent j receives a 0 payment at all outcomes. Thus, consider a scenario where there exist outcomes s_i and s_j such that

$$\tau_i^*(s_i) > 0 \text{ and } \tau_j^*(s_j) > 0.$$

By Lemma 4, it suffices to show that when there exists an outcome $s \in \mathcal{S}$ such that both agents i and j receive a positive payment at this outcome, then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S} \text{ or } \tau_j^*(s) \geq \tau_i^*(s) \text{ for all } s \in \mathcal{S}$$

(or both).

Let $\mathcal{S}_{ij}^*$ be the set of outcomes at which both agents receive a positive payment under contract $\boldsymbol{\tau}^*$. The set $\mathcal{S}_{ij}^*$ is non-empty. We can assume without loss of generality that $\tau_i^*(s) \geq \tau_j^*(s)$ for some outcome $s \in \mathcal{S}_{ij}^*$. We show that then

$$\tau_i^*(s) \geq \tau_j^*(s) \text{ for all } s \in \mathcal{S}.$$

Applying Corollary 1, it holds that

$$|\kappa_i \bar{\kappa}_i| \geq |\kappa_j \bar{\kappa}_j|.$$

Additionally, $\kappa_i \bar{\kappa}_i$ and $\kappa_j \bar{\kappa}_j$ are either both positive or negative. Further applying Corollary 1 to any outcome $s' \in \mathcal{S}_{ij}^*$, the ratio of marginal utilities satisfies

$$\frac{u_i'(\tau_i^*(s'))}{u_j'(\tau_j^*(s'))} = \frac{\kappa_j \bar{\kappa_j}}{\kappa_i \bar{\kappa}_i} \leq 1.$$

This implies that agent i receives a weakly larger payment than agent j under all outcomes in $\mathcal{S}_{ij}^*$, that is,

$$\tau_i^*(s) \geq \tau_j^*(s) \quad \textit{for all } s \in \mathcal{S}_{ij}^*.$$

This ordering on payments holds for outcomes in the set $\mathcal{S} \setminus \mathcal{S}_{ij}^*$ as well. The ordering trivially holds at outcomes where $\tau_j^*(s) = 0$. Consider an outcome s at which $\tau_j^*(s) > 0$ but $\tau_i^*(s) = 0$. We show that such an outcome cannot exist at an optimal contract $\boldsymbol{\tau}^*$. We showed in the proof of Theorem 1 that the first-order condition for the principal is

$$lD\kappa_j\bar{\kappa}_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*) = 0.$$

Since the utility to agent j is strictly increasing, it must hold that

$$lD\kappa_j\bar{\kappa}_j P_s'(Y^*) > 0.$$

Now, consider the following chain of inequalities:

$$\begin{aligned} lD\kappa_i\bar{\kappa}_i u_i'(0)P_s'(Y^*) - P_s(Y^*) &\geq lD\kappa_j\bar{\kappa}_j u_j'(0)P_s'(Y^*) - P_s(Y^*), \\ &> lD\kappa_j\bar{\kappa}_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*), \\ &= 0. \end{aligned}$$

The first inequality holds because $|\kappa_i\bar{\kappa}_i| \geq |\kappa_j\bar{\kappa}_j|$ along with both terms having the same sign and $u_i(\cdot) = u_j(\cdot)$. The second inequality follows from the fact that $u_j(\cdot)$ is strictly concave and $lD\kappa_j\bar{\kappa}_j P_s'(Y^*) > 0$. The left-hand side in the above chain of inequalities is the derivative of the principal's objective in $\tau_i(s)$. The derivative being positive contradicts the optimality of $\boldsymbol{\tau}^*$, so the statement of the proposition holds. $\square$

A.5. **Proof of Corollary 2.** Recall from the proof of Theorem 1 that

$$l\kappa_i\bar{\kappa}_i P_s'(Y^*)u_i'(\tau_i^*(s)) = \frac{P_s(Y^*)}{\sum_{s'\in\mathcal{S}}\left(v_{s'} - \sum_{i\in N}\tau_i(s')\right)P_{s'}'(Y^*)}, \quad \forall s \in \mathcal{S}_i^*.$$

Taking the ratio of the above equation for any pair of outcomes $s_1, s_2 \in \mathcal{S}_i^*$, we obtain

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

The statement is proved.

A.6. **Proof of Proposition 2.** For a risk-neutral agent i, the marginal value of money is constant: $u_i'(\cdot) = a_i$ for some $a_i > 0$. If at the optimal contract, the agent receives a positive payment under two outcomes s_1 and s_2 that occur with positive probability at Y^*, Corollary 2 implies

$$\frac{u_i'(\tau_i^*(s_1))}{u_i'(\tau_i^*(s_2))} = 1 \neq \frac{P_{s_1}(Y^*)}{P_{s_1}'(Y^*)} \cdot \frac{P_{s_2}'(Y^*)}{P_{s_2}(Y^*)}.$$

We assumed the right hand side is not equal to 1 in the statement of Proposition 2. This gives a contradiction, so agent i can receive a positive payment in at most one outcome.

It remains to show all risk-neutral agents receive a payment at the same outcome. The arguments are essentially the same as those used to prove Lemma 4. Consider risk-neutral agents i and j. Suppose that agent i receives a positive payment at outcome s_i while agent j receives a positive payment at a distinct outcome s_j. Applying the arguments in Lemma 4, the inequality obtained for agent i is

$$\frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)} \leq l\kappa_i\bar{\kappa}_i u_i'(0) \leq \frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)}.$$

Similarly, the inequality obtained for agent j is

$$\frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)} \leq l\kappa_i\bar{\kappa}_i u_i'(0) \leq \frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)}.$$

These inequalities imply $\frac{P_{s_i}(Y^*)}{P_{s_i}'(Y^*)} = \frac{P_{s_j}(Y^*)}{P_{s_j}'(Y^*)}$, again contradicting our assumption these values are distinct. Thus, it must be that the risk-neutral agents receive a positive payment at the same outcome.

A.7. **Proof of Proposition 7.** We begin with a lemma, which adapts Lemma 1.

Lemma 5. *Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract with corresponding equilibrium actions $\mathbf{a}^*$ and team performance Y^*. For any agent i, the derivative of team performance in σ_i, evaluated at $\boldsymbol{\sigma}^*$, is*

$$\frac{dY}{d\sigma_i} = l\kappa_i\bar{\kappa}_i \sum_{s\in\mathcal{S}} P_s'(Y^*) v_s u_i'(\sigma_i^* v_s),$$

where l is independent of i and s.

Proof. The steps taken in this proof are exactly the same as those taken in the proof of Lemma 1. We analyze the change in team performance as the equity transfer to an agent is perturbed. Consider an equity payment scheme $\boldsymbol{\tau}^*$ and any agent i. Consider marginally increasing σ_i. The change induced by this perturbation is

$$\frac{\partial Y}{\partial \sigma_i} = \nabla Y(\mathbf{a}^*)^T \cdot \frac{\partial \mathbf{a}^*}{\partial \sigma_i}, \tag{13}$$

where $\mathbf{a}^*$ is the equilibrium action profile for the contract $\boldsymbol{\tau}$. The substance of the proof is analyzing the second term on the right-hand side of (13).

As in Lemma 1, it is without loss to analyze the change in the action of agent i and actions of agents j that take strictly positive actions in profile $\mathbf{a}^*$. The analysis from here on focuses on such agents, overloading notation to represent the actions of these agents by $\mathbf{a}^*$.

We will show that the change in equilibrium actions $\mathbf{a}^*$ as the equity σ_i increases is

$$\frac{\partial \mathbf{a}^*}{\partial \sigma_i} = \mathbf{H}^{-\frac{1}{2}} \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}}\mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1} \mathbf{H}^{-\frac{1}{2}} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s\in\mathcal{S}} P_s'(Y) v_s u_i'(\sigma_i v_s) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \sigma_i} \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}. \tag{14}$$

Consider the equilibrium action profile $\mathbf{a}^*$. For an agent j, the first-order conditions imply a_j^* must solve the equation

$$c_j'(a_j) = \left(\sum_{s\in\mathcal{S}} P_s'(Y) u_j(\sigma_j v_s)\right) \frac{\partial Y}{\partial a_j}. \tag{15}$$

To arrive at (14), let us implicitly differentiate (15) with respect to σ_i. For all $j \neq i$,

$$c_j''(a_j^*)\frac{\partial a_j^*}{\partial \sigma_i} = \left(\sum_{s\in\mathcal{S}} P_s'(Y) u_j\left(\sigma_j v_s\right)\right) \left(\sum_{k=1}^{n} \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \sigma_i}\right) \tag{16}$$

$$+ \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \sigma_i} \cdot \sum_{s\in\mathcal{S}} P_s''(Y) u_j(\sigma_j v_s). \tag{17}$$

Similarly for $j = i$,

$$c_j''(a_j^*)\frac{\partial a_j^*}{\partial \sigma_i} = \left(\sum_{s\in\mathcal{S}} P_s'(Y) u_j(\sigma_j v_s)\right) \left(\sum_{k=1}^{n} \frac{\partial^2 Y}{\partial a_k \partial a_j} \cdot \frac{\partial a_k^*}{\partial \sigma_i}\right) \tag{18}$$

$$+ \frac{\partial Y}{\partial a_j} \sum_{s\in\mathcal{S}} P_s'(Y) v_s u_j'(\sigma_j v_s) + \frac{\partial Y}{\partial a_j} \cdot \frac{\partial Y}{\partial \sigma_i} \sum_{s\in\mathcal{S}} P_s''(Y) u_j(\sigma_j v_s). \tag{19}$$

We can combine (16) and (18) in vector form:

$$\frac{\partial \mathbf{a}^*}{\partial \sigma_i} = \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P_s'(Y) v_s u_i'(\sigma_i v_s) \\ \mathbf{0} \end{bmatrix} + \frac{\partial Y}{\partial \sigma_i} \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

The expression in (14) follows.

Substituting (14) into (13), the change in team performance as the equity payment σ_i increases is

$$\frac{\partial Y}{\partial \sigma_i} = \nabla Y(\mathbf{a}^*)^T \mathbf{H}^{-\frac{1}{2}} \left[\mathbf{I} - \mathbf{H}^{-\frac{1}{2}} \mathbf{U}\mathbf{G}\mathbf{H}^{-\frac{1}{2}}\right]^{-1} \mathbf{H}^{-\frac{1}{2}} \begin{bmatrix} \mathbf{0} \\ \frac{\partial Y}{\partial a_i} \sum_{s \in \mathcal{S}} P_s'(Y) v_s u_i'(\sigma_i v_s) \\ \mathbf{0} \end{bmatrix}$$
$$+ \frac{\partial Y}{\partial \sigma_i} \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

Applying the definitions of κ_i and $\bar{\kappa}_i$, we obtain

$$\frac{\partial Y}{\partial \sigma_i} = \kappa_i \bar{\kappa}_i \sum_{s \in \mathcal{S}} P_s'(Y) v_s u_i'(\sigma_i v_s) + \frac{\partial Y}{\partial \sigma_i} \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}.$$

Rearranging,

$$\frac{\partial Y}{\partial \sigma_i} = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T \left[\mathbf{H} - \mathbf{U}\mathbf{G}\right]^{-1} \mathbf{d}} \cdot \kappa_i \bar{\kappa}_i \sum_{s \in \mathcal{S}} P_s'(Y) v_s u_i'(\sigma_i v_s).$$

Setting $l = \frac{1}{1 - \nabla Y(\mathbf{a}^*)^T [\mathbf{H} - \mathbf{U}\mathbf{G}]^{-1} \mathbf{d}}$ and observing l does not depend on i, we obtain the desired result. $\square$

Proof of Proposition 7. The expected payoff for the principal under equity payment $\boldsymbol{\sigma}$ and corresponding equilibrium actions $\mathbf{a}^*$ is

$$\left(1 - \sum_{i \in N} \sigma_i\right) \sum_{s \in \mathcal{S}} v_s P_s(Y(\mathbf{a}^*)).$$

Suppose $\boldsymbol{\sigma}^*$ is an optimal equity contract inducing equilibrium $\mathbf{a}^*(\boldsymbol{\sigma}^*)$ with team performance Y^*. Consider agent i such that $\sigma_i^* > 0$. Then the first-order condition for σ_i^* implies that

$$\frac{dY}{d\sigma_i}\cdot\underbrace{\left(1-\sum_{i\in N}\sigma_i^*\right)\sum_{s\in\mathcal{S}}v_sP_s'(Y^*)}_{D}=\sum_{s\in\mathcal{S}}v_sP_s(Y^*).$$

The left-hand side is the benefit from increasing σ_i^* while the right-hand side is the expected additional transfer required. Since each outcome occurs with non-zero probability, the summation labeled D is nonzero.

Substituting Lemma 5 in the above equation, we obtain

$$l\kappa_i\bar{\kappa}_i\sum_{s\in\mathcal{S}}P_s'(Y^*)v_su_i'(\sigma_i^*v_s)=\frac{\sum_{s\in\mathcal{S}}v_sP_s(Y^*)}{D},$$
$$\iff \kappa_i\bar{\kappa}_i\sum_{s\in\mathcal{S}}P_s'(Y^*)v_su_i'(\sigma_i^*v_s)=c,$$

where $c=\sum_{s\in\mathcal{S}}v_sP_s(Y^*)/(lD)$. Observing that c is independent of i and the outcome s, the statement of the result follows. □

A.8. **Proof of Proposition 3.** Fixing shares $\boldsymbol{\tau}$ and others' strategies, agent i's expected payoff is strictly concave in his action a_i because $Y(\mathbf{a})$ is linear in a_i, the success probability $P(Y)$ is concave in Y, and the effort cost is strictly convex. So agent i has a unique best response, meaning we need only consider pure-strategy equilibria. Moreover marginal costs at $a_i=0$ are zero while marginal benefits at $a_i=0$ are strictly positive if $\tau_i>0$ and zero if $\tau_i=0$. Since U_i is concave in a_i, this rules out a boundary solution where the first-order condition $\frac{\partial U_i}{\partial a_i}=0$ is not satisfied. So the first-order condition is necessary and sufficient for a best-response.

It follows that the following equations are necessary and sufficient for the vector $\mathbf{a}^*$ to be a Nash equilibrium:

$$[\mathbf{I}-P'(Y^*)\beta\mathbf{TG}]\mathbf{a}^*=P'(Y^*)\boldsymbol{\tau}\text{ and }Y^*=Y(\mathbf{a}^*).$$

Given a constant y such that $P'(y)\beta\rho(\mathbf{TG})\neq 1$, where $\rho(\mathbf{TG})$ is the spectral radius of $\mathbf{TG}$, we can define actions by

$$\mathbf{a}^*(y)=[\mathbf{I}-P'(y)\beta\mathbf{TG}]^{-1}P'(y)\boldsymbol{\tau}.$$

Solutions of the first-order conditions then correspond to solutions to

$$Y(\mathbf{a}^*(y)) = y.$$

The function $Y(\mathbf{a}^*(y))$ is strictly increasing in each coordinate of $\mathbf{a}^*(y)$. We analyze how $\mathbf{a}^*(y)$ changes as y increases. Consider the set

$$y_R := \{y : P'(y)\beta\rho(\mathbf{TG}) < 1\}.$$

Observe that because $P(\cdot)$ is concave, if $y \in y_R$ then $y + \epsilon \in y_R$ for any $\epsilon > 0$. We show that constrained to the set y_R, there exists a unique fixed point to the function $Y(\mathbf{a}^*(y))$. Each coordinate of $\mathbf{a}^*(y)$ is weakly decreasing in y since $P'(\cdot)$ is weakly decreasing (by our assumption $P(\cdot)$ is concave). So $Y(\mathbf{a}^*(y))$ is decreasing, meaning there is at most one solution to $Y(\mathbf{a}^*(y)) = y$. It remains to show a solution to this equation exists.

We claim that we can find y such that $Y(\mathbf{a}^*(y)) \geq y$ and $P'(y)\beta\rho(\mathbf{TG}) < 1$. If $P'(0)\beta\rho(\mathbf{TG}) < 1$, the claim holds with $y = 0$ since $Y(\mathbf{a}^*(0)) \geq 0$. Otherwise, define y_0 by $P'(y_0)\beta\rho(\mathbf{TG}) = 1$. A solution to this equation exists since $P'(y)$ is continuous and converges to zero as $y \to \infty$. Then $Y(\mathbf{a}^*(y)) \to \infty$ as $y \to y_0$ from above, so we have $Y(\mathbf{a}^*(y_0 + \epsilon)) \geq y_0 + \epsilon$ for $\epsilon > 0$ sufficiently small. This completes the proof of the claim.

Since $Y(\mathbf{a}^*(y))$ is decreasing in y, we can also choose y large enough such that $y > Y(\mathbf{a}^*(y))$. Since $Y(\mathbf{a}^*(y))$ is continuous in y, by the intermediate value theorem this function has a fixed point, denoted by y^*. We conclude that there exists a unique solution to $Y(\mathbf{a}^*(y)) = y$ in the set y_R and a corresponding profile $\mathbf{a}^*$ of equilibrium actions.

It remains to show that there does not exist an equilibrium $\mathbf{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\beta\rho(\mathbf{TG}) \geq 1$. The case $\boldsymbol{\tau} = 0$ is immediate as the only equilibrium is $\mathbf{a}^* = 0$. Take $\boldsymbol{\tau}$ not identically zero and suppose there exists an equilibrium $\mathbf{a}^*$ such that $P'(Y^*)\beta\rho(\mathbf{TG}) \geq 1$. It must solve the necessary and sufficient conditions

$$[\mathbf{I} - P'(Y^*)\beta\mathbf{TG}]\mathbf{a}^* = P'(Y^*)\boldsymbol{\tau} \text{ and } Y^* = Y(\mathbf{a}^*). \tag{20}$$

By the Perron-Frobenius theorem,[9] there exists a left-eigenvector $\mathbf{v}$ of the matrix $P'(Y^*)\beta\mathbf{TG}$ such that $\mathbf{v}$ has strictly positive entries. Multiplying the LHS of (20) by the vector v, we get

$$\begin{aligned}\mathbf{v}^T[\mathbf{I} - P'(Y^*)\beta\mathbf{TG}]\mathbf{a}^* &= [1 - P'(Y^*)\beta\rho(\mathbf{TG})]\mathbf{v}^T\mathbf{a}^* \\ &\leq 0,\end{aligned}$$

where the inequality follows from the assumption $P'(Y^*)\beta\rho(\mathbf{TG}) \geq 1$ and the fact that $\mathbf{a}^*$ has strictly positive elements. However, we also compute

$$\begin{aligned}\mathbf{v}^T[\mathbf{I} - P'(Y^*)\beta\mathbf{TG}]\mathbf{a}^* &= \mathbf{v}^T P'(Y^*)\boldsymbol{\tau} \text{ by (20)} \\ &> 0,\end{aligned}$$

where the inequality holds because the entries of $\mathbf{v}$ are all positive and the entries of τ are all non-negative and not identically zero. This is a contradiction, so there does not exist an equilibrium $\mathbf{a}^*$ with corresponding team performance Y^* such that $P'(Y^*)\beta\rho(\mathbf{TG}) \geq 1$. We conclude the equilibrium described above is the unique one.

A.9. **Proof of Lemma 2.** By Theorem 1, we have that

$$\kappa_i\overline{\kappa}_i \text{ is constant across agents } i.$$

Suppose that there exist two agents $i^* \in N$ with $i^* = \arg\min_{k\in N} \kappa_k$ and $j^* \in N$ with $j^* = \arg\max_{k\in N} \kappa_k$ such that $\kappa_{i^*} < \kappa_{j^*}$.[10]

Then we have that, for agent i^*,

$$\kappa_{i^*}\overline{\kappa}_{i^*} < \kappa_{i^*}\kappa_{j^*}\sum_{j\in N}[\mathbf{I} - P'(Y^*)\mathbf{GT}]^{-1}_{i^*j} = (\kappa_{i^*})^2\kappa_{j^*}, \tag{21}$$

[9]For this argument, it is without loss to assume the matrix $\mathbf{TG}$ is irreducible. If not, since $\mathbf{G}$ is symmetric, we can rewrite $\mathbf{TG}$ in a block diagonal form with irreducible blocks. Then $P'(Y^*)\beta\rho(\mathbf{TG})$ must be an eigenvalue of at least one block of the matrix $P'(Y^*)\beta\mathbf{TG}$. We can drop agents in all other blocks and apply the remainder of the argument to this block.

[10]We are grateful to Michael Ostrovsky for suggesting the argument in the next paragraph.

using the maximality of κ_{j^*} among the κ_j and the definitions of $\overline{\kappa}_{i^*}$ and κ_{i^*}. But we similarly have that, for agent j^*,

$$\kappa_{j^*}\overline{\kappa}_{j^*} > \kappa_{j^*}\kappa_{i^*}\sum_{i\in N}[\mathbf{I} - P'(Y^*)\mathbf{G}\mathbf{T}]^{-1}_{j^*i} = \kappa_{i^*}(\kappa_{j^*})^2. \tag{22}$$

Theorem 1 implies that $\kappa_{i^*}\overline{\kappa}_{i^*} = \kappa_{j^*}\overline{\kappa}_{j^*}$ for any two agents i^* and j^*, and so combining (21) and (22) implies

$$(\kappa_{i^*})^2\kappa_{j^*} > \kappa_{i^*}(\kappa_{j^*})^2.$$

This contradicts our assumption $\kappa_{j^*} > \kappa_{i^*}$, so we must have κ_i equal to some constant c_1 for all i in N.

A.10. **Proof of Proposition 5.** Suppose $\boldsymbol{\tau}^*$ is an optimal contract for network $\mathbf{G}$ with equilibrium team performance $Y^*(\mathbf{G}, \boldsymbol{\tau}^*)$. Consider a perturbed network $\widetilde{\boldsymbol{G}}$ generated by increasing edge weight G_{ij} to $G_{ij} + \epsilon$ for some $\epsilon > 0$. We will show that contract $\boldsymbol{\tau}^*$ performs weakly better on network $\widetilde{\boldsymbol{G}}$ than on $\boldsymbol{G}$. Since we will be comparing $\boldsymbol{\tau}^*$ across networks, we suppress the dependence of equilibrium team performance on the contract.

Consider contract $\boldsymbol{\tau}^*$ and network $\widetilde{\boldsymbol{G}}$. We want to show that the equilibrium team performance $Y^*(\widetilde{\mathbf{G}})$ is at least $Y^*(\mathbf{G})$. The equilibrium actions solve

$$\mathbf{a}^*(\widetilde{\boldsymbol{G}}) = P'(Y^*(\widetilde{\boldsymbol{G}}))\left[\mathbf{I} - \beta P'(Y^*(\widetilde{\boldsymbol{G}}))\mathbf{T}^*\widetilde{\mathbf{G}}\right]^{-1}\boldsymbol{\tau}^*.$$

Suppose $Y^*(\widetilde{\boldsymbol{G}}) < Y^*(\boldsymbol{G})$. It follows that $\mathbf{a}^*(\widetilde{\boldsymbol{G}})$ is point-wise strictly greater than $\mathbf{a}^*(\boldsymbol{G})$, because of the concavity of $P(\cdot)$ and the fact that $\widetilde{\boldsymbol{G}}$ is point-wise weakly greater than $\boldsymbol{G}$. However, this is a contradiction to $Y^*(\widetilde{\boldsymbol{G}}) < Y^*(\boldsymbol{G})$ because

$$Y(\mathbf{a}, \boldsymbol{G}) = \sum_i a_i + \frac{\beta}{2}\sum_{i,j} G_{ij}a_ia_j.$$

Thus, we must have $Y^*(\widetilde{\boldsymbol{G}}) \geq Y^*(\boldsymbol{G})$. The profits to the principal under contract $\boldsymbol{\tau}^*$ are thus weakly higher on network $\widetilde{\boldsymbol{G}}$ than on network $\boldsymbol{G}$:

$$\left(1 - \sum_i \tau_i^*\right) P\left(Y^*(\widetilde{\boldsymbol{G}})\right) \geq \left(1 - \sum_i \tau_i^*\right) P\left(Y^*\left(\boldsymbol{G}\right)\right).$$

Finally, the optimal contract for network $\widetilde{\boldsymbol{G}}$ must deliver at least as high a payoff as contract $\boldsymbol{\tau}^*$ does on network $\widetilde{\boldsymbol{G}}$.

A.11. **Proof of Proposition 6.** We can assume without loss of generality that all agents in $\mathbf{G}$ are active under $\boldsymbol{\tau}^*$ (by dropping any inactive agents from the network). Consider a feasible allocation $\boldsymbol{\tau}$ satisfying the balanced neighborhood equity condition $\mathbf{G}\boldsymbol{\tau} = c\mathbf{1}$ and let $s = \sum_i \tau_i \in [0,1]$ be the sum of shares under this allocation. Such an allocation will exist for any $s \in [0,1]$, as $\mathbf{G}$ is the optimal active set and thus $(\mathbf{G}^{-1}\mathbf{1})_i > 0$ for all i in $\mathbf{G}$. The balanced neighborhood equity condition implies that

$$c = \frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}.$$

At any solution which satisfies the balanced equity condition and allocates a fraction s of shares to agents, the team performance

$$Y^* = \mathbf{1}^T\mathbf{a}^* + \frac{\beta}{2}(\mathbf{a}^*)^T\mathbf{G}\mathbf{a}^*$$

can be rewritten as

$$Y^* = \left(\frac{P'(Y^*)}{1-\beta P'(Y^*)c} + \frac{\beta P'(Y^*)^2 c}{2(1-\beta P'(Y^*)c)^2}\right)s. \tag{23}$$

Applying (23) to the linear output setting, we can write the profit for the principal under this allocation as

$$V(s,\beta) = \alpha^2 s(1-s)\left(\frac{1}{1-\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}} + \frac{\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}}{2\left(1-\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}\right)^2}\right).$$

So for a fixed β for which $\boldsymbol{\tau}^*$ is an optimal contract, the total payments under this contract solves the optimization problem

$$V^*(\beta) = \max_{s\in[0,1]} s(1-s)\left(\frac{1}{1-\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}} + \frac{\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}}{2\left(1-\beta\alpha\frac{s}{\mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}}\right)^2}\right).$$

We will characterize the solution to this optimization problem. We define $k^* := \mathbf{1}^T\mathbf{G}^{-1}\mathbf{1}$ and claim that $\beta\alpha < k^*$. We must have $\beta \in \left(0, \frac{1}{\alpha}\frac{1}{\rho(\mathbf{TG})}\right)$ by our assumption that equilibrium team performance is in $[0,\overline{Y}]$. Observe that $c = s/k^*$ is an eigenvalue for the matrix $\mathbf{TG}$ for any $s \in [0,1]$, with right eigenvector $\boldsymbol{\tau}$. Thus we have

$$\frac{s}{k^*} \leq \rho(\mathbf{TG}) < \frac{1}{\beta\alpha}.$$

Choosing $s = 1$ then verifies the claim $\beta\alpha < k^*$.

We now return to the problem of maximizing $V(s,\beta)$. Taking the partial derivative with respect to s, we find

$$\frac{\partial V(s,\beta)}{\partial s} = \frac{k^*\alpha^2\left(-(\beta\alpha)^2 s^3 + 3\beta\alpha k^* s^2 - 4(k^*)^2 s + 2(k^*)^2\right)}{2\left(k^* - \beta\alpha s\right)^3}.$$

It suffices to study the behavior of $V(s,\beta)$ when $s \in [0,1]$. Define

$$p(s,\beta) := -(\beta\alpha)^2 s^3 + 3\beta\alpha k^* s^2 - 4(k^*)^2 s + 2(k^*)^2$$

to be the numerator of $V(s,\beta)$. The partial derivative of $p(s,\beta)$ with respect to s is

$$\frac{\partial p(s,\beta)}{\partial s} = -3(\beta\alpha)^2 s^2 + 6\beta\alpha k^* s - 4(k^*)^2 < -3(\beta\alpha s + k^*)^2.$$

Since the right-hand expression is strictly negative, the function $p(s,\beta)$ is strictly decreasing in $s \in [0,1]$. Thus $p(\cdot,\beta)$ has only one real root for each β.

We claim that this root lies in $\left(\frac{1}{2},1\right)$. At $s = \frac{1}{2}$, we have

$$p\left(\frac{1}{2},\beta\right) = \left(-(\beta\alpha)^2 \cdot \frac{1}{8} + 3\alpha\beta k^* \cdot \frac{1}{4}\right) > 0,$$

for any $\beta\alpha < k^*$. At $s = 1$, we have

$$p(1,\beta) = (k^* - \beta\alpha)(\beta\alpha - 2k^*) < 0,$$

for any $\beta\alpha < k^*$. This proves the claim.

For $s \in [0,1]$, the denominator of $V(s,\beta)$ is strictly positive for any $\beta\alpha < k^*$. So for each β, the sum of shares s at the optimal allocation is characterized by $p(s,\beta) = 0$. We calculate

$$\frac{\partial p(s,\beta)}{\partial \beta} = 3\alpha k^* s^2 - 2\beta\alpha^2 s^3 = \alpha s^2\left(3k^* - 2\beta\alpha s\right) > 0,$$

where the inequality holds for all $s \in (0,1)$. Since $p(s,\beta)$ is strictly decreasing in s for each β, the sum of shares s at the optimal allocation is increasing in β.

Appendix B. Sufficient conditions for positive payments

The balance result in Theorem 1 only applies to agents receiving a positive payment at some outcome. As discussed in Section 5, not all agents necessarily receive positive payments at the optimal contract. In this section, we provide sufficient conditions on the environment which guarantee all agents receive positive payments in all states that would be more likely if team performance increased (that is, all states where $P_s'(Y^*) > 0$).

Assumption 2. *The environment is such that:*

(a) There exists a contract $\boldsymbol{\tau}$ satisfying

$$\tau_i(s) > 0 \quad \text{for some agent } i \text{ and some outcome } s,$$

such that the principal obtains a strictly higher payoff than under the contract where no agent gets paid.

(b) For every agent i, $\lim_{\tau \to 0} u_i'(\tau) = \infty$.

(c) At any optimal contract $\boldsymbol{\tau}^*$, the total effect on team performance satisfies either

$$\bar{\kappa}_i > 0 \text{ for every agent } i \quad \text{or} \quad \bar{\kappa}_i < 0 \text{ for every agent } i.$$

Part (a) ensures that the principal finds it optimal to pay at least one agent in the team. Part (b) is a standard Inada condition for the agent's utility. Part (c) is a homogeneity condition across agents saying that the direction of the effect of every agent on overall team performance is the same.

Under these assumptions, all agents are paid in all states that would become more likely if team performance increased slightly.

Proposition 8. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract with induced team performance Y^*. For any agent i and any state s,*

$$\tau_i^*(s) > 0 \quad \text{if and only if} \quad P_s'(Y^*) > 0.$$

Proof. The proof involves analyzing the derivative of the principal's objective with respect to payments made to the agents. Consider any agent i. As shown in the proof of Theorem 1, the derivative of the principal's objective with respect to $\tau_i(s)$ is given by the expression

$$lD\kappa_i\bar{\kappa}_i u_i'(\tau_i^*(s))P_s'(Y^*) - P_s(Y^*).$$

Forward direction: Consider agent i and let $\mathcal{S}_i^*$ be the set of outcomes at which it receives a positive payment. Then,

$$P_s'(Y^*) > 0, \text{ for all } s \in \mathcal{S}_i^*.$$

The statement of the forward direction is exactly Lemma 3. Note that arguments to prove the lemma did not require an Inada condition.

Backward direction: Any agent i receives a strictly positive payment at all outcomes where $P_s'(Y^*) > 0$, that is,

$$\text{if } P_s'(Y^*) > 0 \quad \text{then } \tau_i^*(s) > 0.$$

To prove the backward direction, first suppose that $\bar{\kappa}_i > 0$ for all agents. By Part (a) of Assumption 2, at the optimal contract $\boldsymbol{\tau}^*$, there exists some agent i receiving a positive payment at some outcome s. Note that from the *forward direction*, we must have $P_s'(Y^*) > 0$. The principal's first-order condition (see proof of Theorem 1), applied to agent i is

$$lD\kappa_i\bar{\kappa}_i u_i'(\tau_i^*(s))P_s'(Y^*) - P_s(Y^*) = 0,$$

which implies $lD > 0$. For any other agent j, the derivative of the principal's objective in $\tau_j(s)$, is given by the expression

$$lD\kappa_j\bar{\kappa}_j u_j'(\tau_j^*(s))P_s'(Y^*) - P_s(Y^*).$$

The formula above was derived in the proof of Theorem 1 by utilizing Lemma 1 which derived the overall effect of a perturbation to payments on team performance. Lemma 1 was stated for any agent receiving a positive payment at some outcome. The result also holds for agents receiving a zero payment at all outcomes, when the perturbation is made at an outcome s at which $P_s'(Y^*) > 0$. [11] By the Inada condition

[11] Recall that Lemma 1 was defined for any agent receiving a positive payment at some outcome. We show that the result also holds for agents receiving zero payments at all outcomes, when the perturbation in payments is made in an outcome where $P_s'(Y^*) > 0$. For any agent i taking action

on the marginal utility function, the observation that $lD\bar{\kappa}_j > 0$ (since $\bar{\kappa}_i > 0$), and the fact that $P'_s(Y^*) > 0$

$$\lim_{\tau_j^*(s)\to 0} lD\kappa_j\bar{\kappa}_j u'_j(\tau_i^*(s))P'_s(Y^*) - P_s(Y^*) > 0.$$

Thus, it cannot be the case that $\tau_j^*(s) = 0$ is optimal. The argument when all $\bar{\kappa}_i$ are negative is essentially the same, with $lD < 0$ instead. □

Appendix C. Existence of equilibria

The main analysis took a pure-strategy equilibrium satisfying certain technical assumptions for granted, and analyzed its comparative statics to derive necessary conditions for contract optimality.

We now provide an environment in which an equilibrium exists for every contract within a large class. The contracts the principal considers satisfy a natural *value-paying* property. Under the following assumption, every contract satisfying this property will have at least one equilibrium.

Assumption 3. *The environment is such that:*

(a) There exists an outcome $s_0 \in \mathcal{S}$ with

$$v_{s_0} = 0,$$

that is, the principal gets no revenue from outcome s_0.

$a_i^* > 0$, the first-order conditions at equilibrium imply that a_i^* solves

$$c'_i(a_i) = \left(\sum_{s'\in\mathcal{S}} P'_{s'}(Y)u_i(\tau_i(s'))\right)\frac{\partial Y}{\partial a_i}.$$

We show the equation must also hold if $a_i^* = 0$. To see this, recall that $a_i^* = 0$ if and only if $\tau_i(s) = 0$ at all outcomes s. Consider the first order-condition when $a_i^* = 0$ and $\tau_i(s) = 0$ for all s:

$$c'_i(0) = \left(\sum_{s'\in\mathcal{S}} P'_{s'}(Y)u_i(0)\right)\frac{\partial Y}{\partial a_i}.$$

The left-hand side is zero because $c'(0) = 0$. The right-hand side is zero since $\sum_{s'\in\mathcal{S}} P'_{s'}(Y) = 0$. So the first-order condition holds in this case as well. Due to the regularity assumption on the environment, it follows that the first-order condition binds when payments are perturbed for such an agent at an outcome where $P'_s(Y^*) > 0$.

(b) *Consider any outcome s not equal to the no-revenue outcome s_0. The probability $P_s(\cdot)$ of outcome s is strictly increasing and strictly concave.*

(c) *The team performance function $Y(\cdot)$ is strictly increasing in agents' actions and has diminishing marginal returns, that is,*

$$\frac{\partial Y}{\partial a_i} > 0 \quad \textit{and} \quad \frac{\partial^2 Y}{\partial a_i^2} \leq 0.$$

The assumption posits that the probability of any outcome with a positive revenue is a strictly increasing and strictly concave function of team performance. An implication of this assumption is that $P_{s_0}(\cdot)$ is strictly decreasing and strictly convex.

The principal optimizes over *value-paying* contracts. These contracts pay agents only when an outcome with strictly positive revenue is observed. A formal definition is provided below.

Definition. *A contract $\boldsymbol{\tau}$ is value-paying if*

$$\tau_i(s_0) = 0 \textit{ for all } i,$$

that is, all agents receive payment 0 *under the no-revenue outcome.*

The definition does not restrict the payments under outcomes which provide nonzero revenue. For example, the total payments to agents can exceed the revenue to the principal. We show that an equilibrium exists for all value-paying contracts.

Proposition 9. *There exists an equilibrium for every value-paying contract.*

Proof. Consider a value-paying contract $\boldsymbol{\tau}$ and the induced game. The action space of agent i is the range $[0, \infty)$. The utility to agent i is given by the expression

$$\mathcal{U}_i(a_i, \mathbf{a}_{-i}) = \sum_{s \in \mathcal{S} \setminus s_0} u_i(\tau_i(s)) P_s\left(Y(a_i, \mathbf{a}_{-i})\right) - c_i(a_i).$$

We will show that there exists an upper bound $\bar{a}_i$ such that agent i only considers actions in the range $[0, \bar{a}_i]$. The utility to agent i for any profile of actions $\mathbf{a}$ can be bounded above by

$$\max_s u_i\left(\tau_i(s)\right) - c_i(a_i). \tag{24}$$

Since the cost of action $c_i(\cdot)$ is continuous, strictly increasing and strictly convex, there exists $\bar{a}_i < \infty$ such that

$$c_i(a_i) \geq \max_s u_i(\tau_i(s)) \quad \text{for all } a_i \geq \bar{a}_i.$$

Thus the utility from choosing $a_i > \bar{a}_i$ is always negative, so any best response for agent i must be contained in $[0, \bar{a}_i]$. So it is without loss to reduce the strategy space of each player to this interval, which is compact.

We can rewrite i's utility under action profile $\mathbf{a}$ as

$$\mathcal{U}_i(\mathbf{a}) = u_i(\tau_i(s_0)) + \sum_{s \in \mathcal{S} \setminus s_0} [u_i(\tau_i(s)) - u_i(\tau_i(s_0))] P_s\left(Y(a_i, \mathbf{a}_{-i})\right) - c_i(a_i). \tag{25}$$

This utility is continuous in vector of actions $\mathbf{a}$ because the team performance $Y(\mathbf{a})$ is continuous in actions and the probabilities $P_s(Y)$ are continuous in team performance. The utility to player i is strictly concave in their own action a_i. To see this, recall that Y is concave in a_i and $P_s(Y)$ is strictly concave in Y, so $P_s\left(Y(a_i, \mathbf{a}_{-i})\right)$ is strictly concave in a_i. Since $c_i(a_i)$ is strictly convex, (25) is a linear combination of strictly concave functions with non-negative weights. Applying the equilibrium existence result in Glicksberg (1952), we obtain a (pure strategy) equilibrium. □

This gives explicit conditions under which the existence assumption in our main result holds.

Appendix D. Comparative statics as the network changes

This section provides two additional comparative statics results in the parametric setting of Section 5. We strengthen a link and ask (1) how the optimal contract changes and (2) how the induced team performance changes.

We first describe how the optimal equity shares vary as the network changes. We write $\frac{\partial}{\partial G_{jk}}$ for the derivative in the weight $G_{jk} = G_{kj}$ of the link between j and k. Recall that given an allocation, we write $\widetilde{\mathbf{G}}$ for the adjacency matrix restricted to active agents.

Proposition 10. *Suppose that under* $\mathbf{G}$ *there is a unique optimal contract* $\boldsymbol{\tau}^*$*, with agents* i*,* j*, and* k *all active. The derivative of agent* i*'s optimal share as we vary the* *weight of the link between j and k is*

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -(\widetilde{\mathbf{G}}^{-1})_{ik}\tau_j^* - (\widetilde{\mathbf{G}}^{-1})_{ij}\tau_k^* + \frac{\partial c}{\partial G_{jk}}\frac{\tau_i^*}{c},$$

where c is the constant from Proposition 4.

The value c is the total equity in each neighborhood, which Proposition 4 shows is constant across agents. The proof is based on the matrix calculus expression

$$\frac{\partial \mathbf{G}(t)^{-1}}{\partial t} = -\mathbf{G}(t)^{-1}\frac{\partial \mathbf{G}(t)}{\partial t}\mathbf{G}(t)^{-1} \tag{26}$$

for the derivative of the inverse of a matrix. The result provides a fairly explicit expression for the impact of changing a link on equity allocations.

Proof of Proposition 10. Proposition 4 tells us that, for all agents such that $\tau_i^* > 0$, we have

$$\boldsymbol{\tau}^* = c\widetilde{\mathbf{G}}^{-1}\mathbf{1}.$$

We will use the matrix calculus expression

$$\frac{\partial \mathbf{G}(t)^{-1}}{\partial t} = -\mathbf{G}(t)^{-1}\frac{\partial \mathbf{G}(t)}{\partial t}\mathbf{G}(t)^{-1}.$$

Taking the derivative with respect to G_{jk}, we have that

$$\frac{\partial \boldsymbol{\tau}^*}{\partial G_{jk}} = -c\widetilde{\mathbf{G}}^{-1}\frac{\partial \widetilde{\mathbf{G}}}{\partial G_{jk}}\widetilde{\mathbf{G}}^{-1}\mathbf{1} + \frac{\partial c}{\partial G_{jk}}\widetilde{\mathbf{G}}^{-1}\mathbf{1}.$$

Analyzing the i^{th} element in this vector gives

$$\frac{\partial \tau_i^*}{\partial G_{jk}} = -c(\widetilde{\mathbf{G}}^{-1}\mathbf{1})_j(\widetilde{\mathbf{G}}^{-1})_{ik} - c(\widetilde{\mathbf{G}}^{-1}\mathbf{1})_k(\widetilde{\mathbf{G}}^{-1})_{ij} + \frac{\partial c}{\partial G_{jk}} \cdot (\widetilde{\mathbf{G}}^{-1}\mathbf{1})_i.$$

The result follows from $\tau_i^* = c(\widetilde{\mathbf{G}}^{-1}\mathbf{1})_i$ and the analogous expressions with indices j and k. □

We next look at how team performance under an optimal contract varies as the network changes. Recall that Y^* denotes the equilibrium team performance under an optimal allocation. Then $\frac{\partial Y^*}{\partial G_{ij}}$ is the change in this team performance as the weight on the link between agent i and j increases.

Proposition 11. *Suppose $\boldsymbol{\tau}^*$ is an optimal contract. Then the change in equilibrium team performance as G_{ij} varies can be expressed as*

$$\frac{\partial Y^*}{\partial G_{ij}} = \tau_i^* \tau_j^* h,$$

where h does not depend on the identities of i or j.

The proposition says that the increase in team performance from strengthening a link is precisely proportional to the product of the payments given to the two agents connected by that link. The proof gives an explicit formula for the quantity h, which depends on the model parameters and the allocation.

The proposition has implications for a designer who can make small changes in the network of complementarities. If the principal could marginally strengthen some links, she would want to focus on links between pairs of agents with large payments.

Proof of Proposition 11. We want to calculate the derivative of the team performance Y^* under the optimal allocation as G_{ij} increases. By the envelope theorem, we can calculate this derivative by holding fixed the allocation $\boldsymbol{\tau}^*$. To do so, we calculate the derivative of the equilibrium team performance Y^* for a given allocation $\boldsymbol{\tau}$ as G_{ij} increases. We will then substitute $\boldsymbol{\tau} = \boldsymbol{\tau}^*$.

Letting $\mathbf{a}^*$ be the equilibrium action profile under the allocation $\boldsymbol{\tau}$, we calculate

$$\begin{aligned}
\frac{\partial Y}{\partial G_{ij}} &= \frac{\partial \mathbf{1}^T \mathbf{a}^* + \frac{\beta}{2} (\mathbf{a}^*)^T \mathbf{G} \mathbf{a}^*}{\partial G_{ij}}, \\
&= \mathbf{1}^T \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \beta (\mathbf{a}^*)^T \mathbf{G} \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \frac{\beta}{2} (\mathbf{a}^*)^T \frac{\partial \mathbf{G}}{\partial G_{ij}} \mathbf{a}^*, \\
&= \left[\mathbf{1}^T + \beta (\mathbf{a}^*)^T \mathbf{G}\right] \frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \beta a_i^* a_j^*.
\end{aligned}$$

The equilibrium action satisfies $\mathbf{a}^* = \beta P'(Y)\mathbf{TGa}^* + P'(Y)\mathbf{T1}$. Thus, we can write

$$\begin{aligned}
\frac{\partial \mathbf{a}^*}{\partial G_{ij}} &= \beta P'(Y)\mathbf{T}\frac{\partial \mathbf{G}}{\partial G_{ij}}\mathbf{a}^* + \beta P'(Y)\mathbf{TG}\frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \left(\beta \mathbf{TGa}^* + \mathbf{T1}\right)\frac{\partial P'(Y)}{\partial G_{ij}}, \\
&= \beta \begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} P'(Y) + \beta P'(Y)\mathbf{TG}\frac{\partial \mathbf{a}^*}{\partial G_{ij}} + \left(\beta \mathbf{TGa}^* + \mathbf{T1}\right) P''(Y)\frac{\partial Y}{\partial G_{ij}}.
\end{aligned}$$

where $\mathbf{T}\frac{\partial \mathbf{G}}{\partial G_{ij}}\mathbf{a}^*$ is a vector with the i^{th} element equal to $\tau_i a_j^*$, the j^{th} element equal to $\tau_j a_i^*$ and the rest of the elements equal to zero. Solving for $\frac{\partial \mathbf{a}^*}{\partial G_{ij}}$ gives

$$\frac{\partial \mathbf{a}^*}{\partial G_{ij}} = \left[\mathbf{I} - \beta P'(Y)\mathbf{TG}\right]^{-1}\left(\beta P'(Y)\begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} + \mathbf{T}\left[\mathbf{1} + \beta \mathbf{Ga}^*\right] P''(Y)\frac{\partial Y}{\partial G_{ij}}\right).$$

Substituting into the expression for $\frac{\partial Y}{\partial G_{ij}}$ gives

$$\begin{aligned}
&\frac{\partial Y}{\partial G_{ij}}\left[1 - \left(\mathbf{1} + \beta \mathbf{Ga}^*\right)^T\left[\mathbf{I} - \beta P'(Y)\mathbf{TG}\right]^{-1}\mathbf{T}\left(\mathbf{1} + \beta \mathbf{Ga}^*\right)P''(Y)\right] \\
&\qquad = \beta P'(Y)\left[\mathbf{1}^T + \beta(\mathbf{a}^*)^T\mathbf{G}\right]\left[I - \beta P'(Y)\mathbf{TG}\right]^{-1}\begin{bmatrix} 0 \\ \tau_i a_j^* \\ 0 \\ \tau_j a_i^* \\ 0 \end{bmatrix} + \beta a_i^* a_j^*.
\end{aligned}$$

We now use the optimality of $\boldsymbol{\tau}$, which implies the equality $\mathbf{a}^* = \boldsymbol{\tau}^*\frac{P'(Y)}{1-\beta c P'(Y)}$ by Proposition 4. Applying this, we obtain

$$\frac{\partial Y^*}{\partial G_{ij}} = \beta \tau_i^* \tau_j^* P'(Y^*)^2 \frac{\left(\frac{2}{(1-\beta c P'(Y^*))^3} + \frac{1}{(1-\beta c P'(Y^*))^2}\right)}{1 - \frac{P''(Y^*)\sum_i \tau_i^*}{(1-\beta c P'(Y^*))^3}}.$$

The right-hand side has the desired form. $\square$