Econ 1033	**Problem set 10**
Prof. Ben Golub	Due: **Thursday, Nov. 19, 2020 11:50pm**

This problem set is worth 100 points. You should choose problems adding up to at least 100 points (write at the top of your problem set which ones you have chosen) and complete them.

Definition 1 (Irreducibility)**.** *For any matrix $\boldsymbol{W}$ with nonnegative entries, we can define a directed graph $\mathcal{G}$ whose edges are all (i, j) with $W_{ij} > 0$ (i.e., there is an edge from i to j if and only if the weight W_{ij} is positive). The definition of $\boldsymbol{W}$ being irreducible is that $\mathcal{G}$ is a strongly connected graph.*[1]

An alternative definition of $\boldsymbol{W}$ being irreducible is: for any nonempty, proper subset $N' \subsetneq \{1, 2, \dots, n\}$ of the nodes, there is an $i \in N'$ and $j \notin N'$ such that $W_{ij} > 0$.

1. (40 points) Consider the DeGroot model with updating matrix $\boldsymbol{W}$. Assume $\boldsymbol{W}$ is irreducible (see above). Assume that there is some i with $W_{ii} > 0$.

a. (10 points) Prove that there is a T so that for all $t \geq T$, all entries of $\boldsymbol{W}^t$ are strictly positive.

b. (5 points) Fix an arbitrary vector of real numbers $\boldsymbol{x}(0) \in \mathbb{R}^n$. Let

$$\boldsymbol{x}(t) = \boldsymbol{W}^t \boldsymbol{x}(0),$$

as in the DeGroot model. Let $\overline{x}(t) = \max_i x_i(t)$ be the maximum entry of $\boldsymbol{x}(t)$ and let $\underline{x}(t) = \min_i x_i(t)$ be the minimum entry of $\boldsymbol{x}(t)$. Show that, for every $t \geq 0$, it holds that $\overline{x}(t+1) \leq \overline{x}(t)$ and $\underline{x}(t+1) \geq \underline{x}(t)$.

c. (10 points) Define

$$d(t) = |\overline{x}(t) - \underline{x}(t)|$$

to be the difference between the maximum and minimum entry of $\boldsymbol{x}(t)$. Show that there exists a positive real number $\delta < 1$ and an integer T' so that, for all $t \geq 0$, we have

$$d(t + T') \leq \delta d(t).$$

d. (5 points) Deduce that, for any $\boldsymbol{x}(0) \in \mathbb{R}^n$, we have that $d(t) \to 0$ as $t \to \infty$. Deduce that, for every $\boldsymbol{x}(0)$, the sequence of vectors $(\boldsymbol{x}(t))_{t=0}^{\infty}$ converges to a vector where every entry is the same.

e. (10 points) Conclude that $\boldsymbol{W}^t$ has a limit as $t \to \infty$. This means that for every entry (i, j), the sequence of real numbers $(\boldsymbol{W}^t)_{ij}$ has a limit as $t \to \infty$.

Congratulations! You have just proved convergence of opinions in the DeGroot model to a consensus (under some assumptions). You have also proved the equivalent mathematical fact that an irreducible, row-stochastic matrix *with at least one positive diagonal entry* has powers that converge to a rank-one limit. (Later in the problem set, you will investigate whether the italicized condition can be weakened.)

[1] Recall a graph is strongly connected if, for every i and j, there is a path from i to j in the graph.

2. (20 points) In the DeGroot model, we fix an arbitrary vector of real numbers $\boldsymbol{x}(0) \in \mathbb{R}^n$ representing initial opinions. We let

$$\boldsymbol{x}(t) = \boldsymbol{W}^t \boldsymbol{x}(0).$$

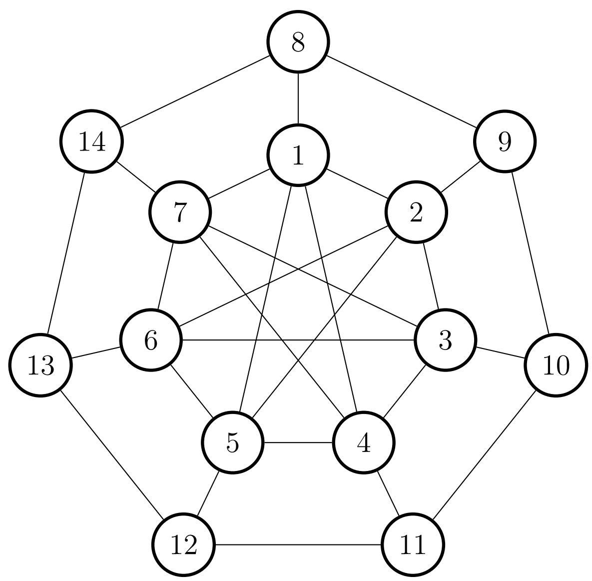
a. (10 points) If $\boldsymbol{W}$ satisfies the conditions of Problem 1, then we know that

$$\boldsymbol{W}^\infty = \begin{bmatrix} - & \boldsymbol{q} & - \\ & \vdots & \\ - & \boldsymbol{q} & - \end{bmatrix},$$

where $\boldsymbol{q}$ is the left-hand Perron vector[2] of $\boldsymbol{W}$. Taking this for granted, Explain why we can interpret q_i as the long-run influence of agent i.

a. (10 points) Work with a two-agent version of the DeGroot model. Can you come up with a 2-by-2 matrix $\boldsymbol{W}$ satisfying all the conditions of Problem 1, yet such that $\boldsymbol{W}^{1000}$ is far from its limit (at difference of at least 0.1 in at least one entry)?

3. (20 points) Consider the following graph.



a. (10 points) Let $\boldsymbol{G}$ be the adjacency matrix of the graph. The vector $\boldsymbol{q}$ of eigenvector centralities in $\boldsymbol{G}$ is the unique nonnegative vector $\boldsymbol{q}$ with entries summing to 1 such that, for some $r > 0$, we have $\boldsymbol{q}\boldsymbol{G} = r\boldsymbol{q}$. Compute the eigenvector centrality of each node in $\boldsymbol{G}$.

b. (10 points) Define $\boldsymbol{W}$ by $W_{ij} = G_{ij}/d_i$, where d_i is the sum of all entries in row i of $\boldsymbol{G}$. The vector $\boldsymbol{q}$ of eigenvector centralities of $\boldsymbol{W}$ is the unique nonnegative vector $\boldsymbol{q}$ with entries summing to 1 such that, for some $r > 0$, we have $\boldsymbol{q}\boldsymbol{W} = r\boldsymbol{q}$. Compute the eigenvector centrality of each node in $\boldsymbol{W}$.

[2] The unique row vector $\boldsymbol{q}$ such that (I) its entries are all positive; (II) $\sum_i q_i = 1$; and (III) $\boldsymbol{q}\boldsymbol{W} = r\boldsymbol{q}$, where r is the spectral radius of $\boldsymbol{W}$.

4. (20 points) Let $\boldsymbol{G}$ be a symmetric matrix with nonnegative entires. Define $\boldsymbol{W}$ by $W_{ij} = G_{ij}/d_i$, where d_i is the sum of all entries in row i of $\boldsymbol{G}$. The vector $\boldsymbol{q}$ of eigenvector centralities of $\boldsymbol{W}$ is the unique nonnegative vector $\boldsymbol{q}$ with entries summing to 1 such that, for some $r > 0$, we have $\boldsymbol{qW} = r\boldsymbol{q}$. Describe the eigenvector centrality of each node in $\boldsymbol{W}$ as explicitly as you can. Hint: the answer will depend on something fairly simple about nodes' positions in $\boldsymbol{G}$.

5. (20 points) Describe two ways in which the DeGroot model is better than canonical Bayesian social learning models ([EK] Chapter 16, Aumann's partitional model) and two ways in which it is worse. Write about 250 words.

6. (20 points) Consider the DeGroot model with updating matrix $\boldsymbol{W}$. Assume throughout this problem that $\boldsymbol{W}$ is irreducible (see definition at the beginning of the problem set).

- **a. (10 points)** Give an example of a DeGroot updating matrix $\boldsymbol{W}$ in which $x_1(t)$, person 1 opinion, fails to converge to a limit as $t \to \infty$.
- **b. (10 points)** Above we saw "$W_{ii} > 0$ for some i" is a sufficient condition for everyone's opinion to converge (assuming irreducibility). Is this condition necessary? Why or why not?