Econ 1033	**Problem set 11**
Prof. Ben Golub	Due: **Thursday, Dec 3, 2020 11:50pm**

This problem set is worth 100 points. You should choose problems adding up to at least 100 points (write at the top of your problem set which ones you have chosen) and complete them.

1. (10 points) In this problem you will do some of the mathematical preliminaries needed to consider the long-run "wisdom of crowds" of the DeGroot model. Suppose that $X_1, \dots, X_n$ are independent random variables, with X_i having mean μ_i and variance σ_i^2.

- **a. (5 points)** Fix any vector $\boldsymbol{q}$ of nonnegative numbers with entries summing to 1. Compute the variance V of the random variable $\sum_{i=1}^n q_i X_i$.
- **b. (5 points)** Let $\overline{q}$ be the maximum entry of $\boldsymbol{q}$ and let $\overline{\sigma}$ be the maximum of the σ_i. Prove that $V \leq \overline{q}\,\overline{\sigma}^2$.

2. (20 points) In this problem you will analyze the long-run "wisdom of crowds" of the DeGroot model. Consider the DeGroot model. Suppose we have a matrix $\boldsymbol{W}$ satisfies conditions guaranteeing convergence to a consensus. Suppose that the initial estimate or opinion of agent i is given by

$$x_i(0) = \theta + \varepsilon_i.$$

The ε_i are independent of each other and are normally distributed with mean 0 and variance σ^2. Let $\boldsymbol{\pi}'$ be the eigenvector centrality vector, so that the consensus is $c = \sum_j \pi_j x_j(0)$.

- **a. (2 points)** Compute the expectation of c. (The randomness is in the draws of initial opinions $x_i(0)$.)
- **b. (3 points)** Compute the variance of c.
- **c. (5 points)** Show that the variance of c is at most $\pi_{\max}\sigma^2$, where $\pi_{\max}$ is defined to be the maximum entry of $\boldsymbol{\pi}$.
- **d. (5 points)** Show that the variance of c is at least $\pi_{\max}^2\sigma^2$.
- **e. (5 points)** Explain the following statement: $\pi_{\max}$ must be small in order for the consensus to reliably approximate the truth θ.
- **f. (5 points)** Now we will think about a sequence $(\boldsymbol{W}(n))_{n=1}^{\infty}$ of influence matrices where $\boldsymbol{\pi}(n)$ is the vector of influence weights in community n. Assume there are n nodes in community n. Give an example of a sequence where $\pi_{\max}(n) \to 0$ as $n \to \infty$, and a different example where $\pi_{\max}(n) \to 1/2$ as $n \to \infty$.

3. (30 points) Consider a complete-information game where each of $n \geq 2$ players (also called agents) $i \in N = \{1, 2, \dots, n\}$ simultaneously selects a real-valued action a_i and receives a payoff $u_i(a_1, a_2, \dots, a_n)$ that depends on everyone's action as follows:

$$u_i(\boldsymbol{a}) = \alpha a_i \sum_j W_{ij} a_j + a_i b_i - \frac{1}{2} a_i^2.$$

If you want a story to go with this, it's in the footnote.[1] Here

$$\alpha \in (0,1), \quad \boldsymbol{W} = (W_{ij})_{i,j\in N}, \text{ and } (b_i)_{i\in N}$$

are constants—parameters of the model that do not depend on $\boldsymbol{a} = (a_i)_{i\in N}$. The matrix $\boldsymbol{W}$ is irreducible,[2] with $W_{ii} = 0$ for every i, and all its entries are nonnegative. Each row of $\boldsymbol{W}$ sums to 1 or less. All the b_i are positive.

a. **(5 points)** Fix any player i. Taking all a_j for $j \neq i$ as given, compute the best-response action of player i. Make sure to check your second-order condition!

b. **(10 points)** Write the condition that everyone is best-responding to everyone's action in matrix notation as $\boldsymbol{a}^* = \cdots$. It should involve the matrix $\boldsymbol{W}$. This is an equation for the Nash equilibrium.

c. **(5 points)** Solve the system you wrote in (b). That is, write a formula for the solution $\boldsymbol{a}^*$ to your system of equations in (b), assuming a solution exists and is unique. There should not be $\boldsymbol{a}^*$ on the right-hand side: only constants that are given as part of the problem setup, including $\boldsymbol{W}$ and $\boldsymbol{b}$.

d. **(10 points)** Suppose players start with initial actions $\boldsymbol{a}(0)$ and do best-response dynamics myopically: at each period, each player sets its action to be its best-response to last period's actions. Describe the dynamics and their limit. You may do so in an example with at least 3 players. Relate the limit to the equilibrium you found above, at least in your chosen example.

4. (20 points) Read pages 681–682 in Meyer on Leontief's Input–Output Economic Model.

Suppose there is only one sector or industry in the economy, so all the matrices involved are 1×1. This one sector produces one good, called widgets. Widgets are valued by the external demander (consumer) as a final good. They can also be used as an intermediate good in the production of widgets. The external demand for widgets is d_1. The total amount of widgets the sector produces is called s_1. To produce 1 unit of widgets, this sector requires a_{11} units of the widgets (an intermediate good purchased from the same industry). Recall all quantities are in units of dollars.

a. **(2 points)** Write the equation relating external demand d_1 for widgets and the total quantity produced of widgets.

b. **(3 points)** Solve this equation for s_1 in terms of d_1.

c. **(5 points)** Suppose the external demand for widgets increases by \$1. How much does the total quantity of widgets produced increase if $a_{11} = 0.5$?

d. **(5 points)** Answer the previous question if $a_{11} = 0.9$?

e. **(5 points)** Explain these counterintuitive answers: why does *total quantity produced* in the economy increase by more than the change in external demand?

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[1] This is a group of students deciding how much to study. The number a_i is the number of hours i spends studying. Studying has a convex cost, proportional to a_i^2. There is a private return b_i per unit of studying – how much i enjoys it; this accounts for the term $a_i b_i$. There is also a part of the return that comes from social spillovers – friends helping each other, or enjoying the process of studying together: the per-unit-time return of studying is therefore increased by $\alpha \sum_j W_{ij} a_j$. That accounts for the first term.

[2] We call a matrix irreducible if the corresponding weighted, directed graph is strongly connected. A 1-by-1 nonnegative matrix is said to be irreducible if its sole entry is positive.

5. (40 points) Read pages 681–682 in Meyer on Leontief's Input–Output Economic Model. Assume throughout that all the entries of $\boldsymbol{A}$ are nonnegative that $\boldsymbol{A}$ is irreducible, and that every column of $\boldsymbol{A}$ sums to less than 1.

a. (5 points) Define

$$\begin{aligned}\boldsymbol{\Lambda} &= (\boldsymbol{I} - \boldsymbol{A})^{-1} \quad \text{and} \\ \widetilde{\boldsymbol{\Lambda}} &= \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \cdots\end{aligned}$$

In this part, you will show that the two are equal. That is, you will show that the formula for $\widetilde{\boldsymbol{\Lambda}}$, assuming it is well-defined (i.e., the series is convergent), gives the inverse of $\boldsymbol{I} - \boldsymbol{A}$. In order to show this, it is enough to show that $\widetilde{\boldsymbol{\Lambda}}$ multiplied by $\boldsymbol{I} - \boldsymbol{A}$ gives the identity matrix. Show this.

b. (5 points) Recall that $\boldsymbol{s} = \boldsymbol{\Lambda d}$, so we can write

$$\begin{aligned}\boldsymbol{s} &= \widetilde{\boldsymbol{\Lambda}}\boldsymbol{d} \\ &= \boldsymbol{d} + \boldsymbol{Ad} + \boldsymbol{A}^2\boldsymbol{d} + \cdots\end{aligned}$$

Now, imagine we change some sectors' public demands, increasing them by a little bit. Give an intuitive description of what the first and second terms ($\boldsymbol{d}$ and $\boldsymbol{Ad}$) in the summation above represent, economically.

c. (10 points) Explain intuitively what the rest of the terms $\boldsymbol{A}^2\boldsymbol{d}$, $\boldsymbol{A}^3\boldsymbol{d}$, ... mean.

d. (5 points) Suppose $n = 4$ (there are four sectors) and

$$\boldsymbol{W} = \begin{bmatrix} \frac{1}{4} & \cdots & \frac{1}{4} \\ \vdots & \ddots & \vdots \\ \frac{1}{4} & \cdots & \frac{1}{4} \end{bmatrix}$$

and $\boldsymbol{A} = \alpha \boldsymbol{W}$, where $\alpha = 0.5$. Suppose the external demand for all sectors increases by \$1. What happens to the sum of all the s_i (total quantity produced) in the economy? What if $\alpha = 0.9$?

e. (3 points) Now go back to the general case, rather than a specific matrix. Recall my notation that $\boldsymbol{\Lambda} = (\boldsymbol{I} - \boldsymbol{A})^{-1}$, so that we can write $\boldsymbol{s} = \boldsymbol{\Lambda d}$. You may use the entries, λ_{ij}, of the matrix $\boldsymbol{\Lambda}$ in your answers going forward.

Suppose sector j experiences a \$1 increase in exogenous demand. What is the increase in sector i's output?

f. (2 points) What is the sum of these numbers across all the i? Call that total b_j. It represents the total effect on supply when j experiences a demand shock.

g. (10 points) Show that the b_j satisfy the following system of equations, where prime denotes transpose:

$$\boldsymbol{b}' = \boldsymbol{1}' + \boldsymbol{b}'\boldsymbol{A} \quad \text{or} \quad b_j = 1 + \sum_i b_i a_{ij} \ \text{ for each } j.$$

h. (5 points) The system of equations in (g) is kind of like the one defining eigenvector centrality, but different. Explain this statement and give an intuitive description of the relationship between b_j and the other b_i.

6. (30 points) Let $\boldsymbol{G}$ be the adjacency matrix of a line with 5 agents (o–o–o–o–o). Construct $\boldsymbol{W}$ by letting each agent place weight 0.8 on herself and allocate weight 0.2 equally among her neighbors.

Suppose there are two issues under discussion, simultaneously, in the same network. Each individual has an opinion on *each* issue at each time $t = 0, 1, 2, \ldots$. They update these opinions according to the DeGroot rule: on issue A they average issue-A opinions, and on issue B they average issue-B opinions.

- **a. (5 points)** Convert the verbal description into a model with equations of opinion updating.
- **b. (5 points)** Choose any initial opinions for all agents on both issues, as long as they are all different. Write code to compute opinions on both issues and to output a table with opinions at $t = 0, 1, 2, \ldots, 10$ and then $t = 10, 15, 20, \ldots, 70$ in steps of 5.
- **c. (10 points)** Make a plot where issue-A opinion is on the horizontal axis and issue-B opinion is on the vertical axis. Plot each person's opinions at times $t = 10, 15, \ldots, 40$, in steps of 5, on this plot. (Again, the initial opinions can be whatever you like; choose them to make the picture look good/interesting in your own assessment.) Make sure the axes dimensions are chosen such that you can see the points as clearly and distinctly as possible. Ideally, also find a way to depict the dynamics of the process.
- **d. (5 points)** Now show the opinions for $t = 40, 45, \ldots, 70$, in steps of 5. Again, choose the zoom so you can see at least some of the points distinctly (you will be zoomed in on a small area!).
- **e. (5 points)** The opinions are converging, as we know they must. Yet if we zoom in on the disagreement remaining at any one time, it should look like opinions are arranging themselves on a line, essentially no matter how you initialize the process. Thus, there is a left-right spectrum, and people who are extremists on one issue also are guaranteed to be extremists on the other! Relate this to real-world political disagreement.

7. (20 points) In [J-HN] Chapter 7, Jackson discusses the double counting issue and why it is hard to do rational inference rather than DeGroot-style naive updating. Give an example of a situation in which people might be good at filtering out "echoes" and avoiding double-counting (as best they can), even though they are learning in a large network. You can be creative with the situation you discuss, who the agents are, what they're learning about, etc., as long as you're specific. Refer to the reasons Jackson gives for the difficulties of optimal filtering and argue against their relevance in your example. Write 250–300 words.

8. (30 points) In [J-HN] Chapter 7, Jackson discusses "rebroadcasting" of news and how that affects incentives to produce valuable, true information in the news industry. Review his argument and critique it at any points where you feel it may not be totally sound. Here's a potential challenge for Jackson's narrative: "rebroadcasting" good tweets and memes is just as easy as repeating the facts of a news story, yet some people and organizations seem to invest heavily in producing good social media content. Is this in tension with Jackson's argument? How can you reconcile the rise of high-quality social media content with the decline of local news? Use all the economic reasoning you can muster to give an illuminating analysis. Write about 350-400 words.