Econ 1033	**Problem set 9**
Prof. Ben Golub	Due: **Thursday, November 12, 2020 at 11:50pm**

This problem set is worth 100 points. You should choose problems adding up to at least 100 points (write at the top of your problem set which ones you have chosen) and complete them.

Reading: Osborne and Rubinstein 5.1–5.3

1. (30 points) In the notation of Osborne and Rubinstein Chapter 5, let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Given two partitions of Ω, say $\mathcal{P}$ and $\mathcal{P}'$, we say $\mathcal{P}$ *refines* (or *is a refinement of*, or *is finer than*) $\mathcal{P}'$ if every set in $\mathcal{P}'$ is a union of sets in $\mathcal{P}$.

a. (5 points) Let

$$\mathcal{P}_1 = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\}$$

$$\mathcal{P}_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}.$$

Is either partition a refinement of the other? Why or why not?

b. (5 points) Suppose we have two partitions $\mathcal{P}_1'$ and $\mathcal{P}_2'$ such that $\mathcal{P}_1'$ refines $\mathcal{P}_2'$. Explain intuitively why we can say that a player with partition $\mathcal{P}_1'$ (i.e., who is told the element of $\mathcal{P}_1'$ containing ω) knows unambiguously weakly more information than a player with with partition $\mathcal{P}_2'$.

c. (5 points) If $\mathcal{P}$ is a refinement of $\mathcal{P}'$ we say that $\mathcal{P}'$ is a coarsening of $\mathcal{P}$. Give an example of a partition with at least two elements (i.e., two different sets) that is a coarsening of $\mathcal{P}_1$ in (a). Explain why the trivial partition $\{\Omega\}$ is always a coarsening of any other partition.

d. (5 points) We define the *meet* of two partitions $\mathcal{P}$ and $\mathcal{P}'$ to be their finest common coarsening: a partition $\mathcal{P}_m$ that is (i) coarser than $\mathcal{P}$ and $\mathcal{P}'$; and (ii) such that there is no partition finer than $\mathcal{P}_m$ (and different from it) that satisfies (i). Write down the meet of $\mathcal{P}_1$ and $\mathcal{P}_2$ in (a).

e. (10 points) Now suppose there is a prior distribution p on Ω, with $p(\omega) \in (0, 1)$ for each ω. Let $E = \{9, 10\}$ be the event that ω is either 9 or 10. Players 1 and 2 privately observe their information and each forms a subjective probability that the event E happened. Give an example (i.e., partitions $\mathcal{P}_1'$ and $\mathcal{P}_2'$ as well as a prior p and a particular realization ω) where the following hold: (i) ω is in E, so that when ω is realized, E occurs; (ii) when ω occurs, player 1 with partition $\mathcal{P}_1'$ has a higher subjective probability of E than player 2 with partition $\mathcal{P}_2'$; (iii) the element of $\mathcal{P}_1'$ containing ω is not a subset of the element of $\mathcal{P}_2'$ containing ω. (The last condition means we cannot say that player 1 is "more informed" than 2, even at this ω).

2. (30 points)

a. (10 points) Obsborne and Rubinstein Exercise 76.1.

b. (10 points) Obsborne and Rubinstein Exercise 76.2. Restrict attention to simple lotteries (i.e., ones that pay 1 on some event and 0 otherwise), and ignore the last sentence of this exercise.

c. (10 points) Explain why this means we cannot have speculative trade in the class example we discussed on November 5.

Note: For the remaining problems, please refer to the following notations and definitions. See here for further references on eigenvalues and eigenvectors.

For an $n \times n$ matrix $\mathbf{A}$:

- If a non-zero (column) vector $\mathbf{x}$ satisfies that $\mathbf{Ax}$ is a *scalar multiple* of $\mathbf{x}$, i.e., $\mathbf{Ax} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an **eigenvector** of $\mathbf{A}$, and the scale factor λ is the **eigenvalue** corresponding to that eigenvector.

  Recall that eigenvalues can be complex for real matrices.

- The set of *distinct* eigenvalues of $\mathbf{A}$, denoted by $\sigma(\mathbf{A})$, is called the **spectrum** of $\mathbf{A}$. That is, if $\lambda_1, \ldots, \lambda_k$ are the (real or complex) distinct eigenvalues of $\mathbf{A}$, then $\sigma(\mathbf{A}) = \{\lambda_1, \ldots, \lambda_k\}$.

- The **spectral radius** of $\mathbf{A}$ is defined as the largest absolute value (modulus) of its eigenvalues, $\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|$. That is, if $\lambda_1, \ldots, \lambda_k$ are the distinct (real or complex) eigenvalues of $\mathbf{A}$, then $\rho(\mathbf{A}) = \max\{|\lambda_1|, \ldots, |\lambda_k|\}$.

3. (20 points)

$$\mathbf{A} = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 8 & 3 \\ 1 & 2 & 9 \end{pmatrix}.$$

a. (5 points) Compute the eigenvalues and eigenvectors for $\mathbf{A}$. Make sure that you include the general form of all eigenvectors. In particular, what are $\sigma(\mathbf{A})$ and $\rho(\mathbf{A})$?

b. (15 points) Verify for this example that $\rho(\mathbf{A}) \in \sigma(\mathbf{A})$. Show that there is a unique eigenvector $\mathbf{p} = (p_j)$ associated with the eigenvalue $\rho(\mathbf{A})$ such that $p_j > 0$ for all j and $\sum_j p_j = 1$. Moreover, show that except for positive multiples of $\mathbf{p}$, there are no other nonnegative eigenvectors for $\mathbf{A}$, regardless of the eigenvalue.[1]

4. (20 points)

$$\mathbf{M} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

a. (5 points) Compute the eigenvalues and eigenvectors for $\mathbf{M}$ and $\mathbf{M}^T$ (the transpose of $\mathbf{M}$). Make sure that you include the general form of all eigenvectors. In particular, verify for this example that $\sigma(\mathbf{M}) = \sigma(\mathbf{M}^T)$.

b. (5 points) Compute $\mathbf{M}^\ell$ numerically (i.e., $\mathbf{M}$ multiplied by itself ℓ times) for $\ell = 5, 10, 15$.

c. (5 points) Did you find any patterns along the sequence? In particular, what does $\lim_{\ell\to\infty} \mathbf{M}^\ell$ seem to be? How is the limit related to your answers in (a)?

d. (5 points) Let $\mathbf{x}(0)$ be an arbitrary column vector in $\mathbb{R}^3$ and let $\boldsymbol{x}(t) = \mathbf{Mx}(t-1)$ for $t \geq 1$. What does $\mathbf{x}(t)$ seem to be converging to as $t \to \infty$? Relate this to your answer in (c).

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[1] This holds in general for **positive matrices**, where $\mathbf{A} = (a_{ij})$each $a_{ij} > 0$. This special eigenvector $\mathbf{p}$ is called the **Perron vector** for $\mathbf{A}$, and the associated eigenvalue $\rho(\mathbf{A})$ is called the **Perron root** for $\mathbf{A}$. These results are special cases of the Perron-Frobenius theorem.