Econ 1033	**Problem set 4**
Prof. Ben Golub	Due: **Wed., Sep 30, 2020 11:50pm**

This problem set is worth 100 points. You should attempt problems totaling at least 100 points, and submit solutions to all parts of those problems. For any subset $\mathcal{P}$ of problems totaling at least 100 points, your $\mathcal{P}$-score is defined to be the fraction of points you receive for problems in $\mathcal{P}$ (i.e., points earned on problems in $\mathcal{P}$ divided by possible point total for problems $\mathcal{P}$, which may be more than 100). We will make your score equal to your $\mathcal{P}$-score where $\mathcal{P}$ (the set of problems that count) is chosen optimally for you.

1. (40 points) Recall the Bass model. Time is discrete, $t = 1, 2, 3, \ldots$. The fraction of "infected" or "innovation-adopting" people is

$$i(t+1) = i(t) + p\,[1 - i(t)] + qi(t)\,[1 - i(t)]\,.$$

Here $p, q \in (0, 1)$ are parameters.

a. **(8 points)** Explain how the formulation of the problem assumes away anything random in the evolution of $i(t)$. When is this a reasonable assumption and what justifies it?

b. **(8 points)** Replace the the "one unit" time increment by dt and explain, informally, how you transform the difference equation above into the differential equation

$$\frac{d}{dt}i(t) = [p + qi(t)]\,[1 - i(t)]\,.$$

where i is now viewed as a function $i : \mathbb{R}_{\geq 0} \to [0, 1]$ defined on the positive real numbers.

c. **(8 points)** Solve the differential equation for an arbitrary given value $i(0) = x$. Explain why the solution you give is the only one.

d. **(8 points)** Taking $i(0) = 0$, give a condition for the solution to be concave in time. What does the solution look like when this condition does not hold?

e. **(8 points)** Digitize the curve for Kentucky in Figure 1 at a time resolution of one year (or better) and find parameters which you consider to be a "best fit" to the curve.

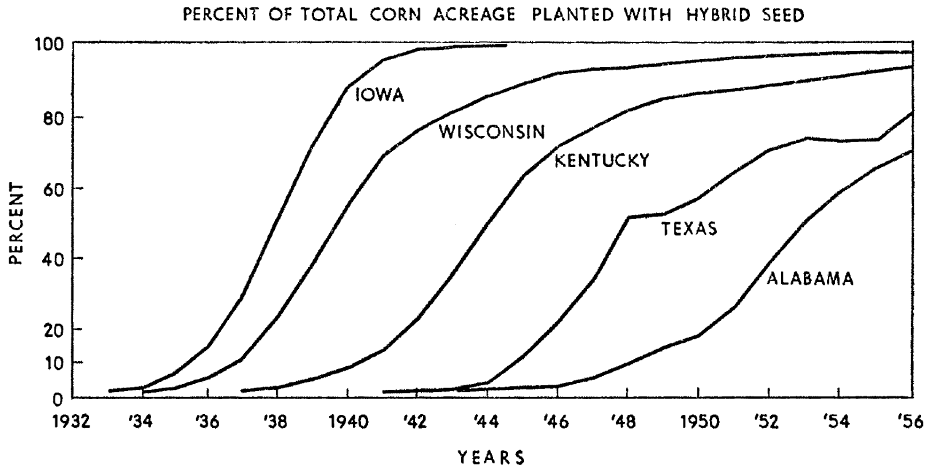

Figure 1

2. (40 points) In this problem we will follow along with a tutorial on the SIR model.

a. (10 points) In part 2 of the tutorial explain the differential equation

$$\frac{dS}{dt} = -bs(t)I(t)$$

for the number of susceptibles and how it leads to

$$\frac{ds}{dt} = -bs(t)i(t).$$

Also explain the equation

$$\frac{dr}{dt} = ki(t)$$

for the fraction of recovereds.

b. (5 points) In part 2 of the tutorial, derive the equation for $\frac{di}{dt}$, explaining your steps.

c. (5 points) Do you think a fixed population (no immigration) is a good assumption for, say, modeling Covid-19? Why or why not?

d. (10 points) Explain under what condition (in terms of s and i) the $i(t)$ curve attains its maximum. Interpret this intuitively.

e. (10 points) Follow the steps in part 5 of the tutorial to explain why

$$\frac{b}{k} = \frac{\ln s_\infty}{s_\infty - 1}.$$

Plot s_∞ as a function of b/k, numerically. Explain intuitively why the curve looks the way it does.

3. (20 points) This exercise is about flattening the curve, herd immunity, etc. Numerically code up the SIR model described in the previous problem in whatever tool you like. (Refer to part 3 of the tutorial.)

- **a. (5 points)** Which single parameter in the model is most affected by physical distancing and mask-wearing ("mitigation measures")? Explain.
- **b. (5 points)** Use your model to show how changing this parameter "flattens the curve." Which curve does it flatten?
- **c. (10 points)** Assume that 2% of all those ever infected die. How do mitigation measures affect the total number of deaths? Use your model's output for various parameters in support of your argument.

4. (40 points) First, some definitions. Fix an undirected graph $G = (V, E)$ without isolated (zero degree) nodes. We will use $N(i)$ to denote the *neighborhood* of a node i—that is, the set of all nodes j that share some edge with i. Define d_i to be the degree of person i, which is also the size of i's neighborhood: $d_i = |N(i)|$.[1] Let $P(d)$ be the fraction of the nodes in the graph with degree d.

- **a. (10 points)** Suppose we pick an edge uniformly at random[2] and then pick either node of that edge with equal probability. Call that node i. Let D be the degree of i; it is a random variable because the node was random. What is the expectation of D? Write your answer in terms of P.
- **b. (10 points)** Prove that the expectation of D is at least as large as the mean of P.
- **c. (20 points)** We make a definition to keep track of how popular i's friends are, on average.

  Definition. *Define δ_i to be the arithmetic mean of the degree of i's friends. That is,*

  $$\delta_i = \frac{\sum_{j \in N(i)} d_j}{d_i}.$$

  Theorem. *For any graph $G = (V, E)$ without isolated nodes, we have*

  $$mean\{d_i\} \leq mean\{\delta_i\},$$

  that is,

  $$\frac{1}{|V|} \sum_{i \in V} d_i \leq \frac{1}{|V|} \sum_{i \in V} \delta_i.$$

  Prove this statement.

5. (20 points) Show that if the number of people at distance d or less from me is at least b^d for some $b > 1$, then the number of steps it takes to reach n people is at most $c \log n$ for some $c > 0$. Write a 200-word explanation of this result at the level of an educated layperson who hasn't taken any math since high school algebra (which he forgot). You will be graded on clarity, correctness, and most of all, ability to actually get the point across to a non-nerd.

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[1] Recall we use the notation $|S|$ to denote the number of elements in the set S.

[2] That is, each edge of the graph is chosen with equal probability.